

A one-loop test for construction of 4D $\mathcal{N} = 4$ SYM from 2D SYM via fuzzy sphere geometry

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Abstract

As a perturbative check of the construction of four-dimensional (4D) $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) from mass deformed $\mathcal{N} = (8, 8)$ SYM on the two-dimensional (2D) lattice, the one-loop effective action for scalar kinetic terms is computed in $\mathcal{N} = 4$ $U(k)$ SYM on $\mathbb{R}^2 \times (\text{fuzzy } S^2)$, which is obtained by expanding 2D $\mathcal{N} = (8, 8)$ $U(N)$ SYM with mass deformation around its fuzzy sphere classical solution. The radius of the fuzzy sphere is proportional to the inverse of the mass. We consider two successive limits: (1) decompactify the fuzzy sphere to a noncommutative (Moyal) plane and (2) turn off the noncommutativity of the Moyal plane. It is straightforward at the classical level to obtain the ordinary $\mathcal{N} = 4$ SYM on \mathbb{R}^4 in the limits, while it is nontrivial at the quantum level. The one-loop effective action for $SU(k)$ sector of the gauge group $U(k)$ coincides with that of the ordinary 4D $\mathcal{N} = 4$ SYM in the above limits. Although “noncommutative anomaly” appears in the overall $U(1)$ sector of the $U(k)$ gauge group, this can be expected to be a gauge artifact not affecting gauge invariant observables.

1 Introduction

The correspondence between four-dimensional (4D) $\mathcal{N} = 4$ $U(N)$ supersymmetric Yang-Mills theory (SYM) and type IIB superstring theory on $AdS_5 \times S^5$ is one of the most typical examples of the AdS/CFT duality conjecture [1, 2, 3]. The correspondence between the 4D $\mathcal{N} = 4$ $U(N)$ SYM in the large N and large 't Hooft coupling limit and the classical gravity limit of the superstring has been supported by numerous pieces of evidence and is almost established. On the other hand, the strong claim of this correspondence between the gauge theory with finite N and quantum superstring theory has been poorly explored and still remains conjectural. This is partly because of a lack of numerical as well as analytical tools beyond perturbative treatment on the gauge theory side. A possible way to overcome this situation is to construct a lattice formulation of 4D $\mathcal{N} = 4$ SYM as its nonperturbative framework. Indeed, in the case of a lower-dimensional version of the duality in the D0-brane system [4], numerical simulations [5, 6, 7, 8, 9, 10, 11] have been developed to provide quantitative nonperturbative tests of the duality beyond the supergravity approximation and bring new aspects into black hole physics. A similar numerical study for the 4D theory will enable an explicit check of this conjecture. Furthermore, if the strong duality conjecture is true, the discretized formulation will provide a nonperturbative description of type IIB superstrings.

Constructing lattice formulations for 4D supersymmetric gauge theories is, however, not straightforward. As an obstacle, it seems impossible to maintain all the supersymmetry on the lattice because of the breakdown of the Leibniz rule [12, 13, 14]. Indeed, a no-go theorem has been proved for constructing a lattice theory with keeping translational invariance, locality, and the Leibniz rule [15]. In order to circumvent this problem, several lattice formulations that keep nilpotent supersymmetries (up to gauge transformations), which do not generate space-time translations, are constructed by applying the so-called orbifolding procedure [16, 17, 18, 19, 20, 21, 22, 23] or topological twists [24, 25, 26, 27, 28, 29, 30, 31] to the discretization. For one- and two-dimensional theories, we can see by a perturbative argument that the continuum limit gives the target theories without any fine-tuning as a result of the exact supersymmetries on the lattice. This has been nonperturbatively shown by numerical simulations as well [32, 33, 34, 35, 36, 37, 38]. However, for 4D supersymmetric gauge theories including $\mathcal{N} = 4$ SYM, the nilpotent supersymmetries are not sufficient to forbid all the relevant operators that prevent the full 4D supersymmetry from restoring in the continuum limit. Therefore, we normally need to tune a number of parameters in taking the continuum limit, which makes it almost impossible to carry

out a numerical simulation ¹. For some progress in 4D $\mathcal{N} = 4$ SYM from 4D lattice, see [41, 42, 43]. Regarding the planar part of the 4D $\mathcal{N} = 4$ SYM, its nonperturbative construction has been given by the plane-wave matrix model [44, 45].

In [46, 47, 48], a new approach to circumvent this issue is proposed for 4D $\mathcal{N} = 2, 4$ SYM theories, where two different discretizations by lattice and matrix [49, 50, 51] are combined. The strategy is as follows. We first construct lattice formulations reminiscent of the “plane-wave matrix string” [50, 52], which are mass deformations of the lattice theories of two-dimensional (2D) $\mathcal{N} = (4, 4)$ and $\mathcal{N} = (8, 8)$ SYM with $U(N)$ gauge group given in [25, 26, 27]. As a result of this deformation, fuzzy sphere configurations realize as classical solutions of the theories. In particular, if we expand field variables around a solution representing k -coincident fuzzy spheres, fluctuations can be regarded as fields of the supersymmetric $U(k)$ SYM on the direct product space of the 2D lattice and the fuzzy sphere. The degrees of freedom of the fuzzy sphere are $n^2 \equiv N^2/k^2$. As mentioned above, the continuum limit of the 2D lattice can be taken without any fine-tuning. In addition, by taking a large- N limit of the original 2D lattice theories with scaling the deformation parameter appropriately, the fuzzy sphere becomes the 2D noncommutative (Moyal) plane \mathbb{R}_Θ^2 . In the rest of this paper, we call this limit the “Moyal limit”. Therefore, if we first take the continuum limit of the lattice theory followed by the large- N limit, we obtain the 4D SYM theories on $\mathbb{R}^2 \times \mathbb{R}_\Theta^2$. In particular, it is argued [53] and shown in the light-cone gauge [54] that the commutative limit ($\mathbb{R}_\Theta^2 \rightarrow \mathbb{R}^2$) of the noncommutative $\mathcal{N} = 4$ SYM theory is continuous. Therefore, numerical simulation of the ordinary 4D $\mathcal{N} = 4$ SYM theory is expected to be possible by this hybrid formulation.

In [46, 47], it is discussed that there appears no radiative correction preventing restoration of the full supersymmetry of the 4D $\mathcal{N} = 4$ theory based on power counting. This argument relies on an assumption that the mass deformation of the 2D theory is soft in 4D theory as well as 2D theory. The deformation parameter M has a positive mass dimension and the deformation is indeed soft in 2D theory, which does not ruin the feature of the 2D lattice theory that no fine-tuning is needed in taking the continuum limit. On the other hand, the situation is more subtle from the viewpoint of the 4D theory, since the same parameter M comes in both the radius of the fuzzy sphere and the noncommutative parameter. Furthermore, it is related to the UV cutoff in the fuzzy sphere directions. Thus, we cannot say that M is an IR deformation in the usual sense. There is still the possibility of unexpected divergences radiatively generated by the so-called UV/IR mixing

¹ The exceptions are $\mathcal{N} = 1$ pure SYM theories in three and four dimensions. The exact parity or chiral symmetry rather than supersymmetry on the lattice plays a key role in restoring the supersymmetry and all the other symmetries in the continuum limit [39, 40].

due to the noncommutativity [55] or via the relation to the UV cutoff of the fuzzy sphere, which may spoil some symmetries to be restored in the 4D continuum theory.

The purpose of this paper is to check if there appear such unexpected divergences by perturbative calculation in Feynman-type gauge fixing at the one-loop order. As mentioned above, the lattice continuum limit can be safely taken even after the deformation, which allows us to start with the deformed theory on the continuum 2D space-time; namely, the deformed 2D $\mathcal{N} = (8, 8)$ SYM theory on \mathbb{R}^2 . We explicitly calculate the one-loop radiative corrections to the one- and two-point functions of bosonic fields, which have larger superficial degrees of UV divergences and are more nontrivial compared to higher-point functions. For the $SU(k)$ part, we will see that there is no unexpected divergence and the 4D rotational ($SO(4)$) symmetry is restored at this order. On the other hand, for the overall $U(1)$ part, there appears a non-trivial correction regarded as the so-called “noncommutative anomaly” [56], which breaks the $SO(4)$ symmetry. However, according to the results for theories on $\mathbb{R}^2 \times \mathbb{R}_\Theta^2$ as well as \mathbb{R}^4 in the light-cone gauge [54, 57], it is considered that this anomaly arises only accompanied with wave function renormalizations. This implies that the anomaly is a gauge artifact and does not affect correlation functions among gauge invariant observables, in which the $SO(4)$ symmetry is expected to be restored.

The organization of this paper is as follows. We present a brief review of the continuum deformed 2D $\mathcal{N} = (8, 8)$ SYM theory in the next section, and expand fields around a classical solution of k -coincident fuzzy spheres by using the so-called fuzzy spherical harmonics in section 3. In section 4, the successive limits leading to the 4D target theory are presented, and propagators are explicitly given for perturbative calculations. In section 5, the one-point functions of bosonic fields are computed at the one-loop order. As a warm up exercise, calculations are presented in some detail. In section 6, we calculate the radiative corrections to the scalar two-point functions at the one-loop level. Since we are interested in the Moyal limit, each Feynman diagram is evaluated in this limit. The use of three theorems proved in appendix D considerably simplifies the computations. In section 7, the one-loop effective action for the scalar kinetic terms is obtained in the successive limits. Section 8 is devoted to a summary of the results obtained so far and discussions of future subjects. In appendix A we give the explicit form of the deformed action in the balanced topological field theory (BTFT) description. In order for this paper to be as self-contained as possible, we derive various useful properties of the fuzzy spherical harmonics in appendix B. Computational details of the one-point functions are collected in appendix C. Appendix D is devoted to proofs of the theorems given in section 6. We give precise relations between the fuzzy sphere and the Moyal plane by taking the Moyal

limit of the fuzzy spherical harmonics in appendix E. To examine the nonperturbative stability of the k -coincident fuzzy sphere solution, tunneling amplitudes to some other vacua are evaluated in appendix F.

2 Review of the deformed 2D $\mathcal{N} = (8, 8)$ SYM

In this section we briefly review the deformed 2D $\mathcal{N} = (8, 8)$ SYM given in [46]. Let us start with $U(N)$ $\mathcal{N} = (8, 8)$ SYM on \mathbb{R}^2 in a Euclidean signature. This theory has gauge fields A_μ ($\mu = 1, 2$), eight scalar fields, and sixteen fermionic fields. For convenience, we express the scalar fields as (X_i, X_a) ($i = 3, \dots, 7$, $a = 8, 9, 10$) and the fermionic fields as $\psi_{r\alpha}$ ($r = 1, \dots, 8$, $\alpha = \pm\frac{1}{2}$). Each field is represented by an $N \times N$ matrix. Note that the scalars (X_8, X_9, X_{10}) and eight pairs of the fermions $\psi_r \equiv (\psi_{r+\frac{1}{2}}, \psi_{r-\frac{1}{2}})$ form a triplet and doublets of an $SU(2)_R$ subgroup of the $SO(8)$ R-symmetry, respectively, and the other fields are $SU(2)_R$ singlets. In this expression, the action can be written as

$$S_0 = \frac{1}{g_{2d}^2} \int d^2x \text{Tr} \left\{ F_{12}^2 + (\mathcal{D}_\mu X_i)^2 + (\mathcal{D}_\mu X_a)^2 - \frac{1}{2} [X_i, X_j]^2 - \frac{1}{2} [X_a, X_b]^2 - [X_a, X_i]^2 \right. \\ \left. + i\bar{\psi}_{r\alpha}(\hat{\gamma}_\mu)_{rs}\mathcal{D}_\mu\psi_{s\alpha} - \bar{\psi}_{r\alpha}(\hat{\gamma}_i)_{rs}[X_i, \psi_{s\alpha}] - \bar{\psi}_{r\alpha}(\hat{\sigma}_a)_{\alpha\beta}[X_a, \psi_{r\beta}] \right\}, \quad (2.1)$$

where $\mathcal{D}_\mu = \partial_\mu + i[A_\mu, \cdot]$ are covariant derivatives for adjoint fields, $F_{12} = \partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2]$ is the gauge field strength, $\hat{\gamma}_I = (\hat{\gamma}_\mu, \hat{\gamma}_i)$ ($I = (\mu, i) = 1, \dots, 7$) are 8×8 matrices satisfying $\{\hat{\gamma}_I, \hat{\gamma}_J\} = -2\delta_{IJ}$, $\hat{\sigma}_a$ are Pauli matrices,

$$\hat{\sigma}_8 = \sigma_1, \quad \hat{\sigma}_9 = \sigma_2, \quad \hat{\sigma}_{10} = \sigma_3, \quad (2.2)$$

and $\bar{\psi}_{r\alpha} \equiv \psi_{r\beta}(i\sigma_2)_{\beta\alpha}$. In this paper, we use the following convention for $\hat{\gamma}_I$ matrices:

$$\hat{\gamma}_1 \equiv \begin{bmatrix} & & \mathbb{1}_2 & \\ & & & \mathbb{1}_2 \\ -\mathbb{1}_2 & & & \\ & & -\mathbb{1}_2 & \end{bmatrix}, \quad \hat{\gamma}_2 \equiv \begin{bmatrix} & & -i\sigma_2 & \\ & & & i\sigma_2 \\ -i\sigma_2 & & & \\ & & i\sigma_2 & \end{bmatrix}, \\ \hat{\gamma}_3 \equiv \begin{bmatrix} & & & i\sigma_2 \\ & & i\sigma_2 & \\ & i\sigma_2 & & \\ i\sigma_2 & & & \end{bmatrix}, \quad \hat{\gamma}_4 \equiv \begin{bmatrix} & & & \mathbb{1}_2 \\ & & -\mathbb{1}_2 & \\ & \mathbb{1}_2 & & \\ -\mathbb{1}_2 & & & \end{bmatrix},$$

$$\begin{aligned}
\hat{\gamma}_5 &\equiv \begin{bmatrix} & \sigma_1 & \\ -\sigma_1 & & \\ & & -\sigma_1 \\ & \sigma_1 & \end{bmatrix}, & \hat{\gamma}_6 &\equiv \begin{bmatrix} & \sigma_3 & \\ -\sigma_3 & & \\ & & -\sigma_3 \\ & \sigma_3 & \end{bmatrix}, \\
\hat{\gamma}_7 &\equiv \begin{bmatrix} -i\sigma_2 & & & \\ & -i\sigma_2 & & \\ & & i\sigma_2 & \\ & & & i\sigma_2 \end{bmatrix}.
\end{aligned} \tag{2.3}$$

Here and in what follows, blank elements in matrices mean null.

Next we deform the action by adding mass terms to scalars and fermions as well as the so-called Myers term [51]:

$$S = S_0 + S_M \tag{2.4}$$

with

$$\begin{aligned}
S_M = \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \Big\{ & \frac{M^2}{9} X_a^2 + i \frac{M}{3} \epsilon_{abc} X_a [X_b, X_c] + \frac{2M^2}{81} X_i^2 - m_r \bar{\psi}_{r\kappa} \psi_{r\kappa} \\
& - i \frac{4M}{9} X_7 (F_{12} + i[X_3, X_4]) \Big\},
\end{aligned} \tag{2.5}$$

where ϵ_{abc} is a third-rank antisymmetric tensor defined by $\epsilon_{8,9,10} = 1$ and

$$m_r = \left(\frac{M}{9}, \frac{M}{9}, \frac{M}{3}, \frac{M}{3}, \frac{2M}{9}, \frac{2M}{9}, \frac{2M}{9}, \frac{M}{3} \right). \tag{2.6}$$

We have also introduced the pure imaginary term $\operatorname{Tr} \left(-i \frac{4M}{9} X_7 (F_{12} + i[X_3, X_4]) \right)$ in (2.5). This is motivated by the deformation of the BTFT description of 2D $\mathcal{N} = (8, 8)$ SYM, which manifestly preserves two supercharges [46]². In the BTFT description, we rename the scalar fields as

$$\begin{aligned}
B_1(x) &\equiv -X_5(x), & B_2(x) &\equiv X_6(x), & B_3(x) &\equiv X_7(x), \\
C(x) &\equiv 2X_8(x), & \phi_{\pm}(x) &\equiv X_9(x) \pm iX_{10}(x),
\end{aligned} \tag{2.7}$$

and write the fermions as

$$\begin{aligned}
\psi_1 &\equiv \begin{pmatrix} \rho_{-3} \\ \rho_{+3} \end{pmatrix}, & \psi_2 &\equiv \begin{pmatrix} \rho_{-4} \\ \rho_{+4} \end{pmatrix}, & \psi_3 &\equiv \begin{pmatrix} \psi_{-2} \\ \psi_{+2} \end{pmatrix}, & \psi_4 &\equiv \begin{pmatrix} \psi_{-1} \\ \psi_{+1} \end{pmatrix}, \\
\psi_5 &\equiv \begin{pmatrix} \chi_{-1} \\ \chi_{+1} \end{pmatrix}, & \psi_6 &\equiv \begin{pmatrix} \chi_{-2} \\ \chi_{+2} \end{pmatrix}, & \psi_7 &\equiv \begin{pmatrix} \chi_{-3} \\ \chi_{+3} \end{pmatrix}, & \psi_8 &\equiv \begin{pmatrix} -\frac{\eta_-}{2} \\ \frac{\eta_+}{2} \end{pmatrix}.
\end{aligned} \tag{2.8}$$

² The other fourteen supercharges are softly broken by the deformation.

Then we can show that the deformed action (2.4) is invariant under the transformation:

$$\begin{aligned}
Q_{\pm} A_{\mu} &= \psi_{\pm\mu}, \quad Q_{\pm} \psi_{\pm\mu} = \pm i D_{\mu} \phi_{\pm}, \quad Q_{\mp} \psi_{\pm\mu} = \frac{i}{2} D_{\mu} C \mp \tilde{H}_{\mu}, \\
Q_{\pm} \tilde{H}_{\mu} &= [\phi_{\pm}, \psi_{\mp\mu}] \mp \frac{1}{2} [C, \psi_{\pm\mu}] \mp \frac{i}{2} D_{\mu} \eta_{\pm} + \frac{M}{3} \psi_{\pm\mu}, \\
Q_{\pm} X_{\underline{i}} &= \rho_{\pm\underline{i}}, \quad Q_{\pm} \rho_{\pm\underline{i}} = \mp [X_{\underline{i}}, \phi_{\pm}], \quad Q_{\mp} \rho_{\pm\underline{i}} = -\frac{1}{2} [X_{\underline{i}}, C] \mp \tilde{h}_{\underline{i}}, \\
Q_{\pm} \tilde{h}_{\underline{i}} &= [\phi_{\pm}, \rho_{\mp\underline{i}}] \mp \frac{1}{2} [C, \rho_{\pm\underline{i}}] \pm \frac{1}{2} [X_{\underline{i}}, \eta_{\pm}] + \frac{M}{3} \rho_{\pm\underline{i}}, \\
Q_{\pm} B_{\mathbf{A}} &= \chi_{\pm\mathbf{A}}, \quad Q_{\pm} \chi_{\pm\mathbf{A}} = \pm [\phi_{\pm}, B_{\mathbf{A}}], \quad Q_{\mp} \chi_{\pm\mathbf{A}} = -\frac{1}{2} [B_{\mathbf{A}}, C] \mp H_{\mathbf{A}}, \\
Q_{\pm} H_{\mathbf{A}} &= [\phi_{\pm}, \chi_{\mp\mathbf{A}}] \pm \frac{1}{2} [B_{\mathbf{A}}, \eta_{\pm}] \mp \frac{1}{2} [C, \chi_{\pm\mathbf{A}}] + \frac{M}{3} \chi_{\pm\mathbf{A}}, \\
Q_{\pm} C &= \eta_{\pm}, \quad Q_{\pm} \eta_{\pm} = \pm [\phi_{\pm}, C] + \frac{2M}{3} \phi_{\pm}, \quad Q_{\mp} \eta_{\pm} = \mp [\phi_{\mp}, \phi_{\mp}] \pm \frac{M}{3} C, \\
Q_{\pm} \phi_{\pm} &= 0, \quad Q_{\mp} \phi_{\pm} = \mp \eta_{\pm}.
\end{aligned} \tag{2.9}$$

Here, the range of the index i ($= 3, \dots, 7$) has been decomposed into $\underline{i} = 3, 4$ and $\mathbf{A} = 1, 2, 3$. The supercharges Q_{\pm} satisfy the anti-commutation relations,

$$Q_{+}^2 = \frac{M}{3} J_{++}, \quad Q_{-}^2 = -\frac{M}{3} J_{--}, \quad \{Q_{+}, Q_{-}\} = -\frac{M}{3} J_0, \tag{2.10}$$

up to gauge transformations, where J_0 , J_{++} and J_{--} are generators of $SU(2)_R$ symmetry acting on fields as

$$\begin{aligned}
J_{++} &= \int d^2x \left[\psi_{+\mu}^{\alpha}(x) \frac{\delta}{\delta \psi_{-\mu}^{\alpha}(x)} + \rho_{+\underline{i}}^{\alpha}(x) \frac{\delta}{\delta \rho_{-\underline{i}}^{\alpha}(x)} + \chi_{+\mathbf{A}}^{\alpha}(x) \frac{\delta}{\delta \chi_{-\mathbf{A}}^{\alpha}(x)} - \eta_{+}^{\alpha}(x) \frac{\delta}{\delta \eta_{-}^{\alpha}(x)} \right. \\
&\quad \left. + 2\phi_{+}^{\alpha}(x) \frac{\delta}{\delta C^{\alpha}(x)} - C^{\alpha}(x) \frac{\delta}{\delta \phi_{-}^{\alpha}(x)} \right], \\
J_{--} &= \int d^2x \left[\psi_{-\mu}^{\alpha}(x) \frac{\delta}{\delta \psi_{+\mu}^{\alpha}(x)} + \rho_{-\underline{i}}^{\alpha}(x) \frac{\delta}{\delta \rho_{+\underline{i}}^{\alpha}(x)} + \chi_{-\mathbf{A}}^{\alpha}(x) \frac{\delta}{\delta \chi_{+\mathbf{A}}^{\alpha}(x)} - \eta_{-}^{\alpha}(x) \frac{\delta}{\delta \eta_{+}^{\alpha}(x)} \right. \\
&\quad \left. - 2\phi_{-}^{\alpha}(x) \frac{\delta}{\delta C^{\alpha}(x)} + C^{\alpha}(x) \frac{\delta}{\delta \phi_{+}^{\alpha}(x)} \right], \\
J_0 &= \int d^2x \left[\psi_{+\mu}^{\alpha}(x) \frac{\delta}{\delta \psi_{+\mu}^{\alpha}(x)} - \psi_{-\mu}^{\alpha}(x) \frac{\delta}{\delta \psi_{-\mu}^{\alpha}(x)} + \rho_{+\underline{i}}^{\alpha}(x) \frac{\delta}{\delta \rho_{+\underline{i}}^{\alpha}(x)} - \rho_{-\underline{i}}^{\alpha}(x) \frac{\delta}{\delta \rho_{-\underline{i}}^{\alpha}(x)} \right. \\
&\quad + \chi_{+\mathbf{A}}^{\alpha}(x) \frac{\delta}{\delta \chi_{+\mathbf{A}}^{\alpha}(x)} - \chi_{-\mathbf{A}}^{\alpha}(x) \frac{\delta}{\delta \chi_{-\mathbf{A}}^{\alpha}(x)} + \eta_{+}^{\alpha}(x) \frac{\delta}{\delta \eta_{+}^{\alpha}(x)} - \eta_{-}^{\alpha}(x) \frac{\delta}{\delta \eta_{-}^{\alpha}(x)} \\
&\quad \left. + 2\phi_{+}^{\alpha}(x) \frac{\delta}{\delta \phi_{+}^{\alpha}(x)} - 2\phi_{-}^{\alpha}(x) \frac{\delta}{\delta \phi_{-}^{\alpha}(x)} \right]
\end{aligned} \tag{2.11}$$

(α is an index for the gauge group generators), which satisfy

$$[J_{++}, J_{--}] = J_0, \quad [J_0, J_{\pm\pm}] = \pm 2J_{\pm\pm}. \quad (2.12)$$

The eigenvalues of J_0 are ± 1 for the fermions with the index \pm , ± 2 for ϕ_{\pm} , and zero for the other bosonic fields. As mentioned above, ϕ_{\pm} and C form an $SU(2)_R$ triplet and each pair of $(\psi_{+\mu}, \psi_{-\mu})$, $(\rho_{+\underline{i}}, \rho_{-\underline{i}})$, $(\chi_{+\mathbf{A}}, \chi_{-\mathbf{A}})$, $(\eta_+, -\eta_-)$, and (Q_+, Q_-) forms a doublet.

The invariance of the action under Q_{\pm} -transformations is most easily seen by the fact that the deformed action (2.4) is recast as

$$S = \left(Q_+ Q_- - \frac{M}{3} \right) \mathcal{F}, \quad (2.13)$$

where

$$\begin{aligned} \mathcal{F} = \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \Big\{ & -iB_{\mathbf{A}}\Phi_{\mathbf{A}} - \frac{1}{3}\epsilon_{\mathbf{ABC}}B_{\mathbf{A}}[B_{\mathbf{B}}, B_{\mathbf{C}}] - \frac{M}{9}B_{\mathbf{A}}^2 - \frac{2M}{9}X_{\underline{i}}^2 \\ & - \psi_{+\mu}\psi_{-\mu} - \rho_{+\underline{i}}\rho_{-\underline{i}} - \chi_{+\mathbf{A}}\chi_{-\mathbf{A}} - \frac{1}{4}\eta_+\eta_- \Big\} \end{aligned} \quad (2.14)$$

with $\Phi_1 = 2(-D_1X_3 - D_2X_4)$, $\Phi_2 = 2(-D_1X_4 + D_2X_3)$, $\Phi_3 = 2(-F_{12} + i[X_3, X_4])$. In fact, the invariance is shown by (2.10) and the commutation relation between $(J_{\pm\pm}, J_0)$ and Q_{\pm} :

$$[J_{\pm\pm}, Q_{\pm}] = 0, \quad [J_{\pm\pm}, Q_{\mp}] = Q_{\pm}, \quad [J_0, Q_{\pm}] = \pm Q_{\pm}. \quad (2.15)$$

In appendix A we give the explicit form of the action in the BTFT description.

3 Fuzzy sphere solution and mode expansion

In this section, we expand the action around a particular supersymmetry preserving solution (a k -coincident fuzzy sphere solution) and explicitly give its mode expansion, which is convenient to carry out perturbative calculations in the subsequent sections.

3.1 Action around fuzzy sphere solution

As a result of the deformation by (2.5), the theory has fuzzy sphere solutions as minima of the action ($S = 0$) preserving Q_{\pm} supersymmetries:

$$X_a(x) = \frac{M}{3}L_a, \quad (\text{other fields}) = 0, \quad (3.1)$$

where L_a belong to an N -dimensional (not necessary irreducible) representation of the $SU(2)$ -algebra satisfying

$$[L_a, L_b] = i\epsilon_{abc}L_c. \quad (3.2)$$

Among a lot of possible solutions, we consider k -coincident fuzzy S^2 with the size n described by

$$L_a = L_a^{(n)} \otimes \mathbb{1}_k, \quad (3.3)$$

where $L_a^{(n)}$ are the n -dimensional irreducible representation of $su(2)$ with $N = nk$. The fields X_a are expanded around the solution (3.3) as

$$X_a(x) = \frac{M}{3} L_a + \tilde{X}_a(x). \quad (3.4)$$

Introducing the “field strength”

$$\begin{aligned} F_{ab} &\equiv \frac{M}{3} \left(i[L_a, \tilde{X}_b] - i[L_b, \tilde{X}_a] + \epsilon_{abc} \tilde{X}_c \right) + i[\tilde{X}_a, \tilde{X}_b], \\ F_{\mu a} &\equiv \partial_\mu \tilde{X}_a - i \frac{M}{3} [L_a, A_\mu] + i[A_\mu, \tilde{X}_a], \end{aligned} \quad (3.5)$$

and the “covariant derivatives”

$$\mathcal{D}_a \equiv i \frac{M}{3} [L_a, \cdot] + i[\tilde{X}_a, \cdot] \quad (3.6)$$

recasts the bosonic and fermionic parts of the action (2.4) : $S = S_b + S_f$ as

$$\begin{aligned} S_b = \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \left\{ F_{12}^2 + \frac{1}{2} F_{ab}^2 + F_{\mu a}^2 + (\mathcal{D}_\mu X_i)^2 + (\mathcal{D}_a X_i)^2 - \frac{1}{2} [X_i, X_j]^2 \right. \\ \left. + \frac{2M^2}{81} X_i^2 - i \frac{4M}{9} X_7 (F_{12} + i[X_3, X_4]) \right\}, \end{aligned} \quad (3.7)$$

$$\begin{aligned} S_f = \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \left\{ i \bar{\psi}_{r\alpha} (\hat{\gamma}_\mu)_{rs} \mathcal{D}_\mu \psi_{s\alpha} + i \bar{\psi}_{r\alpha} (\hat{\sigma}_a)_{\alpha\beta} \mathcal{D}_a \psi_{r\beta} \right. \\ \left. - \bar{\psi}_{r\alpha} (\hat{\gamma}_i)_{rs} [X_i, \psi_{s\alpha}] - m_r \bar{\psi}_{r\alpha} \psi_{r\alpha} \right\}. \end{aligned} \quad (3.8)$$

Upon perturbative calculations, we fix the gauge to a Feynman-type gauge:

$$F(A, \tilde{X}) \equiv \partial_\mu A_\mu + i \frac{M}{3} [L_a, \tilde{X}_a] = 0. \quad (3.9)$$

Under the gauge transformation

$$\delta A_\nu = \mathcal{D}_\nu c, \quad \delta \tilde{X}_a = i \frac{M}{3} [L_a, c] + i[\tilde{X}_a, c] \quad (3.10)$$

(the latter is obtained from $\delta X_a = -i[c, X_a]$), $F(A, \tilde{X})$ changes as

$$\delta F(A, \tilde{X}) = \partial_\mu \mathcal{D}_\mu c - \frac{M^2}{9} [L_a, [L_a, c]] - \frac{M}{3} \left([L_a, [\tilde{X}_a, c]] \right). \quad (3.11)$$

Thus, the gauge fixing terms to the Feynman-type gauge and the associated Faddeev-Popov ghost terms are given by

$$\begin{aligned}
S_{\text{GF}} &= \frac{2}{g_{2d}^2} \int d^2x \text{Tr} \left\{ \frac{1}{2} F(A, \tilde{X})^2 + \bar{c} \delta F(A, \tilde{X}) \right\} \\
&= \frac{1}{g_{2d}^2} \int d^2x \text{Tr} \left\{ (\partial_\mu A_\mu)^2 - \left(\frac{M}{3} \right)^2 [L_a, \tilde{X}_a]^2 + i \frac{2M}{3} (\partial_\mu A_\mu) [L_a, \tilde{X}_a] \right. \\
&\quad \left. - 2 (\partial_\mu \bar{c}) (\mathcal{D}_\mu c) - i \frac{2M}{3} [L_a, \bar{c}] (\mathcal{D}_a c) \right\}, \tag{3.12}
\end{aligned}$$

where c and \bar{c} are ghost and anti-ghost fields, respectively. The action after the gauge fixing is given by the summation of (3.7), (3.8), and (3.12):

$$S_{\text{tot}} = S_b + S_f + S_{\text{GF}}. \tag{3.13}$$

3.2 Fuzzy spherical harmonics and mode expansion

As discussed in [46], the derivatives and the gauge fields along directions of the fuzzy S^2 with the radius $R = \frac{3}{M}$ are given by two linearly independent combinations of $i \frac{M}{3} [L_a, \cdot]$ ($a = 8, 9, 10$) and by the two corresponding combinations of \tilde{X}_a , respectively. Integration over the fuzzy sphere corresponds to taking a partial trace over the n dimensions in the total trace “Tr”. This means that the action (3.7) and (3.8) can be regarded as the action of mass-deformed 4D $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^2 \times (\text{fuzzy } S^2)$. In doing perturbative calculations, it is convenient to expand all the fields in the action (3.13) by the momentum basis on \mathbb{R}^2 and by the basis of fuzzy spherical harmonics on the fuzzy S^2 . The fuzzy spherical harmonics are $n \times n$ matrices, and their definitions and relevant properties are presented in appendix B.

The fields $A_\mu(x)$, $X_i(x)$, $c(x)$, and $\bar{c}(x)$ are expanded by using the scalar fuzzy spherical harmonics $\hat{Y}_{Jm}^{(jj)}$ as

$$\begin{aligned}
A_\mu(x) &= \sum_{J=0}^{2j} \sum_{m=-J}^J \int \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \hat{Y}_{Jm}^{(jj)} \otimes a_{\mu, Jm}(p), \\
X_i(x) &= \sum_{J=0}^{2j} \sum_{m=-J}^J \int \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \hat{Y}_{Jm}^{(jj)} \otimes x_{i, Jm}(p), \\
c(x) &= \sum_{J=0}^{2j} \sum_{m=-J}^J \int \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \hat{Y}_{Jm}^{(jj)} \otimes c_{Jm}(p), \\
\bar{c}(x) &= \sum_{J=0}^{2j} \sum_{m=-J}^J \int \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \hat{Y}_{Jm}^{(jj)} \otimes \bar{c}_{Jm}(p), \tag{3.14}
\end{aligned}$$

where $n = 2j + 1$, and $a_{\mu, Jm}(p)$, $x_{i, Jm}(p)$, $c_{Jm}(p)$, and $\bar{c}_{Jm}(p)$ are $k \times k$ matrices.

$\vec{Y}(x) \equiv \left(\tilde{X}_9(x), \tilde{X}_{10}(x), \tilde{X}_8(x) \right)^T$ is expanded by the vector fuzzy spherical harmonics $\vec{Y}_{Jm(jj)}^\rho$ as

$$\vec{Y}(x) = \sum_{\rho=-1}^1 \sum_{J=\delta_{\rho,0}}^{2j-\delta_{\rho,-1}} \sum_{m=-Q}^Q \int \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \vec{Y}_{Jm(jj)}^\rho \otimes y_{Jm,\rho}(p), \quad (3.15)$$

where $Q \equiv J + \delta_{\rho,1}$, and $y_{Jm,\rho}(p)$ are $k \times k$ matrices. Note that m runs from $-J - 1$ to $J + 1$ for $\rho = 1$, and J runs from 0 to $2j - 1$ for $\rho = -1$. Also, $J = 1, \dots, 2j$ for $\rho = 0$.

The fermions $\psi_{r\alpha}(x)$ are expanded by the spinor fuzzy spherical harmonics,

$$\hat{Y}_{Jm(jj)}^\kappa = \begin{pmatrix} \hat{Y}_{Jm(jj)\alpha=\frac{1}{2}}^\kappa \\ \hat{Y}_{Jm(jj)\alpha=-\frac{1}{2}}^\kappa \end{pmatrix}, \quad (3.16)$$

as

$$(\sigma_1)_{\alpha\beta} \psi_{r\beta}(x) = \sum_{\kappa=\pm 1} \sum_{J=\frac{1}{2}\delta_{\kappa,-1}}^{2j-\frac{1}{2}\delta_{\kappa,-1}} \sum_{m=-U}^U \int \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \hat{Y}_{Jm(jj)\alpha}^\kappa \otimes \psi_{Jm,\kappa}(p)_r, \quad (3.17)$$

where $U \equiv J + \frac{1}{2}\delta_{\kappa,1}$, and $\psi_{Jm,\kappa}(p)_r$ are $k \times k$ matrices. Note that $m = -J - \frac{1}{2}, \dots, J + \frac{1}{2}$ and $J = 0, \dots, 2j$ for $\kappa = 1$, while $m = -J, \dots, J$ and $J = \frac{1}{2}, \dots, 2j - \frac{1}{2}$ for $\kappa = -1$.

We also define the vertex coefficients

$$\hat{C}_{J_2 m_2(jj) J_3 m_3(jj)}^{J_1 m_1(jj)} \equiv \frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{J_1 m_1}^{(jj)} \right)^\dagger \hat{Y}_{J_2 m_2}^{(jj)} \hat{Y}_{J_3 m_3}^{(jj)} \right\}, \quad (3.18)$$

$$\hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{Jm(jj)} \equiv \sum_{i=1}^3 \frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{Jm}^{(jj)} \right)^\dagger \hat{Y}_{J_1 m_1(jj)i}^{\rho_1} \hat{Y}_{J_2 m_2(jj)i}^{\rho_2} \right\}, \quad (3.19)$$

$$\begin{aligned} & \hat{\mathcal{E}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2 J_3 m_3(jj) \rho_3} \\ & \equiv \sum_{i', j', k'=1}^3 \epsilon_{i' j' k'} \frac{1}{n} \text{tr}_n \left\{ \hat{Y}_{J_1 m_1(jj)i'}^{\rho_1} \hat{Y}_{J_2 m_2(jj)j'}^{\rho_2} \hat{Y}_{J_3 m_3(jj)k'}^{\rho_3} \right\}, \end{aligned} \quad (3.20)$$

$$\hat{\mathcal{F}}_{J_2 m_2(jj) \kappa_2 Jm(jj)}^{J_1 m_1(jj) \kappa_1} \equiv \sum_{\alpha=-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{J_1 m_1(jj)\alpha}^{\kappa_1} \right)^\dagger \hat{Y}_{J_2 m_2(jj)\alpha}^{\kappa_2} \hat{Y}_{Jm(jj)i}^{(jj)} \right\}, \quad (3.21)$$

$$\begin{aligned} & \hat{\mathcal{G}}_{J_2 m_2(jj) \kappa_2 Jm(jj) \rho}^{J_1 m_1(jj) \kappa_1} \\ & \equiv \sum_{\alpha, \beta=-\frac{1}{2}}^{\frac{1}{2}} \sum_{i=1}^3 \sigma_{\alpha\beta}^i \frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{J_1 m_1(jj)\alpha}^{\kappa_1} \right)^\dagger \hat{Y}_{J_2 m_2(jj)\beta}^{\kappa_2} \hat{Y}_{Jm(jj)i}^\rho \right\}. \end{aligned} \quad (3.22)$$

Substituting (3.14), (3.15), and (3.17) into the action (3.13) and using these vertices, we obtain the action with respect to the modes both for the momentum in the 2D flat directions and the angular momentum in the fuzzy S^2 directions. We write it as

$$S_{\text{tot}} = S_{2,B} + S_{2,F} + S_{3,B} + S_{3,F} + S_4, \quad (3.23)$$

where $S_{2,B}$ and $S_{2,F}$ are bosonic and fermionic kinetic terms, $S_{3,B}$ and $S_{3,F}$ denote bosonic and fermionic 3-point interaction terms, and S_4 represents (bosonic) 4-point interaction terms. The interaction terms are further decomposed as

$$\begin{aligned} S_{3,B} &= S_{3,B}^C + S_{3,B}^{\mathcal{D}} + S_{3,B}^{\mathcal{E}}, \\ S_{3,F} &= S_{3,F}^{\mathcal{F}} + S_{3,F}^{\mathcal{G}}, \\ S_4 &= S_4^{CC} + S_4^{C\mathcal{D}} + S_4^{\mathcal{D}\mathcal{D}} + S_4^{\mathcal{E}\mathcal{E}}, \end{aligned} \quad (3.24)$$

corresponding the vertex coefficients that appear. The explicit form of the kinetic terms reads

$$\begin{aligned} S_{2,B} &= \frac{n}{g_{2d}^2} \int \frac{d^2 q}{(2\pi)^2} \sum_{j=0}^{2j} \sum_{m=-J}^J (-1)^m \\ &\quad \times \text{tr}_k \left[\left(q^2 + \frac{M^2}{9} J(J+1) \right) a_{\nu, J-m}(-q) a_{\nu, Jm}(q) \right. \\ &\quad + \left(q^2 + \frac{2M^2}{81} + \frac{M^2}{9} J(J+1) \right) x_{i, J-m}(-q) x_{i, Jm}(q) \\ &\quad + \frac{4M}{9} x_{7, J-m}(-q) (q_1 a_{2, Jm}(q) - q_2 a_{1, Jm}(q)) \\ &\quad \left. - 2 \left(q^2 + \frac{M^2}{9} J(J+1) \right) \bar{c}_{J-m}(-q) c_{Jm}(q) \right] \\ &\quad + \frac{n}{g_{2d}^2} \int \frac{d^2 q}{(2\pi)^2} \sum_{j=0}^{2j} \sum_{m=-J-1}^{J+1} (-1)^{m-1} \text{tr}_k \left[\left(q^2 + \frac{M^2}{9} (J+1)^2 \right) y_{J-m, \rho=1}(-q) y_{Jm, \rho=1}(q) \right] \\ &\quad + \frac{n}{g_{2d}^2} \int \frac{d^2 q}{(2\pi)^2} \sum_{j=0}^{2j-1} \sum_{m=-J}^J (-1)^{m-1} \text{tr}_k \left[\left(q^2 + \frac{M^2}{9} (J+1)^2 \right) y_{J-m, \rho=-1}(-q) y_{Jm, \rho=-1}(q) \right] \\ &\quad + \frac{n}{g_{2d}^2} \int \frac{d^2 q}{(2\pi)^2} \sum_{j=1}^{2j} \sum_{m=-J}^J (-1)^{m-1} \text{tr}_k \left[\left(q^2 + \frac{M^2}{9} J(J+1) \right) y_{J-m, \rho=0}(-q) y_{Jm, \rho=0}(q) \right], \end{aligned} \quad (3.25)$$

$$S_{2,F} = \frac{n}{g_{2d}^2} \int \frac{d^2 q}{(2\pi)^2} \sum_{j=0}^{2j} \sum_{m=-J-\frac{1}{2}}^{J+\frac{1}{2}} (-1)^{-m-\frac{1}{2}} \text{tr}_k \left[\Psi_{J-m, \kappa=1}(-q)^T \hat{D}_{J, \kappa=1}(q) \Psi_{Jm, \kappa=1}(q) \right]$$

$$+ \frac{n}{g_{2d}^2} \int \frac{d^2 q}{(2\pi)^2} \sum_{J=\frac{1}{2}}^{2j-\frac{1}{2}} \sum_{m=-J}^J (-1)^{-m+\frac{1}{2}} \text{tr}_k \left[\Psi_{J-m, \kappa=-1}(-q)^T \hat{D}_{J, \kappa=-1}(q) \Psi_{Jm, \kappa=-1}(q) \right], \quad (3.26)$$

with $q^2 \equiv q_\mu q_\mu$,

$$\Psi_{Jm, \kappa}(q) \equiv (\psi_{Jm, \kappa}(q)_1, \dots, \psi_{Jm, \kappa}(q)_8)^T, \quad (3.27)$$

$$\hat{D}_{J, \kappa=1}(q) \equiv \begin{bmatrix} & & & & -q_1 & q_2 & & \\ & & & & -q_2 & -q_1 & & \\ & & & & & & -q_1 & -q_2 \\ & & & & & & q_2 & -q_1 \\ q_1 & q_2 & & & & & & \\ -q_2 & q_1 & & & & & & \\ & & q_1 & -q_2 & & & & \\ & & q_2 & q_1 & & & & \end{bmatrix} - \frac{M}{3} \text{diag} \left(J + \frac{1}{3}, J + \frac{1}{3}, J + 1, J + 1, J + \frac{2}{3}, J + \frac{2}{3}, J + \frac{2}{3}, J + 1 \right) \quad (3.28)$$

and

$$\hat{D}_{J, \kappa=-1}(q) \equiv \begin{bmatrix} & & & & -q_1 & q_2 & & \\ & & & & -q_2 & -q_1 & & \\ & & & & & & -q_1 & -q_2 \\ & & & & & & q_2 & -q_1 \\ q_1 & q_2 & & & & & & \\ -q_2 & q_1 & & & & & & \\ & & q_1 & -q_2 & & & & \\ & & q_2 & q_1 & & & & \end{bmatrix} + \frac{M}{3} \text{diag} \left(J + \frac{7}{6}, J + \frac{7}{6}, J + \frac{1}{2}, J + \frac{1}{2}, J + \frac{5}{6}, J + \frac{5}{6}, J + \frac{5}{6}, J + \frac{1}{2} \right). \quad (3.29)$$

Here “ tr_k ” denotes the k -dimensional trace acting on the modes. The 3-point interaction terms are expressed as

$$S_{3,B}^C = \frac{n}{g_{2d}^2} \int \frac{d^2 p}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \frac{d^2 r}{(2\pi)^2} (2\pi)^2 \delta^2(p + q + r) \\ \times \sum_{J, J', J''=0}^{2j} \sum_{m=-J}^J \sum_{m'=-J'}^{J'} \sum_{m''=-J''}^{J''} (-1)^m \hat{C}_{J' m' (jj)}^{J-m (jj)} J'' m'' (jj)$$

$$\begin{aligned}
& \times \text{tr}_k \left[2(p_1 - r_1) a_{1,Jm}(q) a_{2,J'm'}(p) a_{2,J''m''}(r) \right. \\
& + 2(p_2 - r_2) a_{2,Jm}(q) a_{1,J'm'}(p) a_{1,J''m''}(r) \\
& + 2(p_\mu - r_\mu) a_{\mu,Jm}(q) x_{i,J'm'}(p) x_{i,J''m''}(r) \\
& + \frac{4M}{9} x_{7,Jm}(p) a_{1,J'm'}(q) a_{2,J''m''}(r) - \frac{4M}{9} x_{7,Jm}(p) a_{2,J'm'}(q) a_{1,J''m''}(r) \\
& + \frac{4M}{9} x_{7,Jm}(p) x_{3,J'm'}(q) x_{4,J''m''}(r) - \frac{4M}{9} x_{7,Jm}(p) x_{4,J'm'}(q) x_{3,J''m''}(r) \\
& \left. + 2p_\mu \bar{c}_{Jm}(p) a_{\mu,J'm'}(q) c_{J''m''}(r) - 2p_\mu \bar{c}_{Jm}(p) c_{J'm'}(q) a_{\mu,J''m''}(r) \right], \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
S_{3,B}^{\mathcal{D}} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r) \\
& \times \sum_{J=0}^{2j} \sum_{m=-J}^J \sum_{\rho', \rho''=-1}^1 \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{J''=\delta_{\rho'',0}}^{2j-\delta_{\rho'',-1}} \sum_{m'=-Q'}^{Q'} \sum_{m''=-Q''}^{Q''} (-1)^m \hat{\mathcal{D}}_{J'm'(jj)\rho'J''m''(jj)\rho''}^{J-m(jj)} \\
& \times \text{tr}_k [2(p_\mu - r_\mu) a_{\mu,Jm}(q) y_{J'm',\rho'}(p) y_{J''m'',\rho''}(r)] \\
& + \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r) \\
& \times \sum_{J,J'=0}^{2j} \sum_{m=-J}^J \sum_{m'=-J'}^{J'} \sum_{\rho''=-1}^1 \sum_{J''=\delta_{\rho'',0}}^{2j-\delta_{\rho'',-1}} \sum_{m''=-Q''}^{Q''} (-1)^{m'} \frac{M}{3} \sqrt{J(J+1)} \\
& \times \left\{ \hat{\mathcal{D}}_{J''m''(jj)\rho''Jm(jj)0}^{J'-m'(jj)} \text{tr}_k [2a_{\mu,Jm}(p) a_{\mu,J'm'}(q) y_{J''m'',\rho''}(r) \right. \\
& \quad + 2x_{i,Jm}(p) x_{i,J'm'}(q) y_{J''m'',\rho''}(r) \\
& \quad \left. - 2\bar{c}_{Jm}(p) c_{J'm'}(q) y_{J''m'',\rho''}(r) \right] \\
& + \hat{\mathcal{D}}_{Jm(jj)0J''m''(jj)\rho''}^{J'-m'(jj)} \text{tr}_k [-2a_{\mu,J'm'}(p) a_{\mu,Jm}(q) y_{J''m'',\rho''}(r) \\
& \quad - 2x_{i,J'm'}(p) x_{i,Jm}(q) y_{J''m'',\rho''}(r) \\
& \quad \left. + 2\bar{c}_{Jm}(p) y_{J''m'',\rho''}(q) c_{J'm'}(r) \right] \Big\}, \tag{3.31}
\end{aligned}$$

$$\begin{aligned}
S_{3,B}^{\mathcal{E}} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r) \\
& \times \sum_{\rho,\rho',\rho''=-1}^1 \sum_{J=\delta_{\rho,0}}^{2j-\delta_{\rho,-1}} \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{J''=\delta_{\rho'',0}}^{2j-\delta_{\rho'',-1}} \sum_{m=-Q}^Q \sum_{m'=-Q'}^{Q'} \sum_{m''=-Q''}^{Q''} i \frac{2M}{3} \rho(J+1)
\end{aligned}$$

$$\times \hat{\mathcal{E}}_{Jm(jj)\rho J'm'(jj)\rho' J''m''(jj)\rho''} \text{tr}_k [y_{Jm,\rho}(p) y_{J'm',\rho'}(q) y_{J''m'',\rho''}(r)], \quad (3.32)$$

$$\begin{aligned} S_{3,F}^{\mathcal{F}} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r) \\ &\times \sum_{\kappa,\kappa''=\pm 1} \sum_{J=\frac{1}{2}\delta_{\kappa,-1}}^{2j-\frac{1}{2}\delta_{\kappa,-1}} \sum_{J''=\frac{1}{2}\delta_{\kappa'',-1}}^{2j-\frac{1}{2}\delta_{\kappa'',-1}} \sum_{m=-U}^U \sum_{m''=-U''}^{U''} \sum_{J'=0}^{2j} \sum_{m'=-J'}^{J'} (-1)^{m-\frac{1}{2}\kappa} \hat{\mathcal{F}}_{J''m''(jj)\kappa'' J'm'(jj)}^{J-m(jj)\kappa} \\ &\times \text{tr}_k [2\Psi_{Jm,\kappa}(p)^T \tilde{\gamma}_\mu \Psi_{J''m'',\kappa''}(r) a_{\mu,J'm'}(q) + 2\Psi_{Jm,\kappa}(p)^T \tilde{\gamma}_i \Psi_{J''m'',\kappa''}(r) x_{i,J'm'}(q)], \end{aligned} \quad (3.33)$$

$$\begin{aligned} S_{3,F}^{\mathcal{G}} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r) \\ &\times \sum_{\kappa,\kappa''=\pm 1} \sum_{J=\frac{1}{2}\delta_{\kappa,-1}}^{2j-\frac{1}{2}\delta_{\kappa,-1}} \sum_{J''=\frac{1}{2}\delta_{\kappa'',-1}}^{2j-\frac{1}{2}\delta_{\kappa'',-1}} \sum_{m=-U}^U \sum_{m''=-U''}^{U''} \sum_{\rho'=-1}^1 \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{m'=-Q'}^{Q'} (-1)^{m-\frac{1}{2}\kappa} \\ &\times \hat{\mathcal{G}}_{J''m''(jj)\kappa'' J'm'(jj)\rho'}^{J-m(jj)\kappa} \text{tr}_k [2\Psi_{Jm,\kappa}(p)^T \Psi_{J''m'',\kappa''}(r) y_{J'm',\rho'}(q)]. \end{aligned} \quad (3.34)$$

Finally, the 4-point interaction terms are given by

$$\begin{aligned} S_4^{CC} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \frac{d^2\ell}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r+\ell) \\ &\times \sum_{J,J',J'',J''',J_1=0}^{2j} \sum_{m=-J}^J \sum_{m'=-J'}^{J'} \sum_{m''=-J''}^{J''} \sum_{m'''=-J'''}^{J'''} \sum_{m_1=-J_1}^{J_1} \\ &\times (-1)^{m_1} \hat{C}_{Jm(jj) J'm'(jj)}^{J_1 m_1(jj)} \hat{C}_{J''m''(jj) J'''m'''(jj)}^{J_1 - m_1(jj)} \\ &\times \text{tr}_k [-2a_{1,Jm}(p) a_{2,J'm'}(q) a_{1,J''m''}(r) a_{2,J'''m'''}(\ell) \\ &\quad + 2a_{1,Jm}(p) a_{1,J'm'}(q) a_{2,J''m''}(r) a_{2,J'''m'''}(\ell) \\ &\quad - 2a_{\mu,Jm}(p) x_{i,J'm'}(q) a_{\mu,J''m''}(r) x_{i,J'''m'''}(\ell) \\ &\quad + 2a_{\mu,Jm}(p) a_{\mu,J'm'}(q) x_{i,J''m''}(r) x_{i,J'''m'''}(\ell) \\ &\quad - x_{i,Jm}(p) x_{j,J'm'}(q) x_{i,J''m''}(r) x_{j,J'''m'''}(\ell) \\ &\quad + x_{i,Jm}(p) x_{i,J'm'}(q) x_{j,J''m''}(r) x_{j,J'''m'''}(\ell)], \end{aligned} \quad (3.35)$$

$$\begin{aligned} S_4^{CD} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \frac{d^2\ell}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r+\ell) \\ &\times \sum_{J,J'',J_1=0}^{2j} \sum_{m=-J}^J \sum_{m''=-J''}^{J''} \sum_{m_1=-J_1}^{J_1} \sum_{\rho',\rho'''=-1}^1 \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{J'''=\delta_{\rho''',0}}^{2j-\delta_{\rho''',-1}} \sum_{m'=-Q'}^{Q'} \sum_{m'''=-Q'''}^{Q'''} \end{aligned}$$

$$\begin{aligned}
& \times (-1)^{m_1} \hat{C}_{Jm(jj)J''m''(jj)}^{J_1m_1(jj)} \hat{\mathcal{D}}_{J'm'(jj)\rho'J'''m'''(jj)\rho'''}^{J_1-m_1(jj)} \\
& \times \text{tr}_k [2a_{\mu,Jm}(p) a_{\mu,J''m''}(r) y_{J'm',\rho'}(q) y_{J'''m''',\rho'''}(\ell) \\
& + 2x_{i,Jm}(p) x_{i,J''m''}(r) y_{J'm',\rho'}(q) y_{J'''m''',\rho'''}(\ell)], \tag{3.36}
\end{aligned}$$

$$\begin{aligned}
S_4^{\mathcal{D}\mathcal{D}} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \frac{d^2\ell}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r+\ell) \\
& \times \sum_{J,J''=0}^{2j} \sum_{m=-J}^J \sum_{m''=-J''}^{J''} \sum_{\rho',\rho''',\rho_1=-1}^1 \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{J'''=\delta_{\rho''',0}}^{2j-\delta_{\rho''',-1}} \sum_{J_1=\delta_{\rho_1,0}}^{2j-\delta_{\rho_1,-1}} \sum_{m'=-Q'}^{Q'} \sum_{m'''=-Q'''}^{Q'''} \sum_{m_1=-Q_1}^{Q_1} \\
& \times (-1)^{m+m''+m_1} \hat{\mathcal{D}}_{J'm'(jj)\rho'J_1m_1(jj)\rho_1}^{J-m(jj)} \hat{\mathcal{D}}_{J'''m'''(jj)\rho''J_1-m_1(jj)\rho_1}^{J''-m''(jj)} \\
& \times \text{tr}_k [2a_{\mu,Jm}(p) y_{J'm',\rho'}(q) a_{\mu,J''m''}(r) y_{J'''m''',\rho'''}(\ell) \\
& + 2x_{i,Jm}(p) y_{J'm',\rho'}(q) x_{i,J''m''}(r) y_{J'''m''',\rho'''}(\ell)], \tag{3.37}
\end{aligned}$$

$$\begin{aligned}
S_4^{\mathcal{E}\mathcal{E}} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \frac{d^2\ell}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r+\ell) \sum_{\rho,\rho',\rho'',\rho''',\rho_1=-1}^1 \\
& \times \sum_{J=\delta_{\rho,0}}^{2j-\delta_{\rho,-1}} \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{J''=\delta_{\rho'',0}}^{2j-\delta_{\rho'',-1}} \sum_{J'''=\delta_{\rho''',0}}^{2j-\delta_{\rho''',-1}} \sum_{J_1=\delta_{\rho_1,0}}^{2j-\delta_{\rho_1,-1}} \sum_{m=-Q}^Q \sum_{m'=-Q'}^{Q'} \sum_{m''=-Q''}^{Q''} \sum_{m'''=-Q'''}^{Q'''} \sum_{m_1=-Q_1}^{Q_1} \\
& \times (-1)^{-m_1} \hat{\mathcal{E}}_{Jm(jj)\rho J'J''m''(jj)\rho'J_1m_1(jj)\rho_1} \hat{\mathcal{E}}_{J'''m'''(jj)\rho''J'''m'''(jj)\rho''J_1-m_1(jj)\rho_1} \\
& \times \text{tr}_k [y_{Jm,\rho}(p) y_{J'm',\rho'}(q) y_{J''m'',\rho''}(r) y_{J'''m''',\rho'''}(\ell)]. \tag{3.38}
\end{aligned}$$

For later convenience, we present another expression for $S_4^{C\mathcal{D}}$ in terms of only the \mathcal{D} coefficients rather than the C and \mathcal{D} coefficients:

$$\begin{aligned}
S_4^{C\mathcal{D}} &= \frac{n}{g_{2d}^2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \frac{d^2\ell}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+r+\ell) \\
& \times \sum_{J,J''=0}^{2j} \sum_{m=-J}^J \sum_{m''=-J''}^{J''} \sum_{\rho',\rho''',\rho_1=-1}^1 \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{J'''=\delta_{\rho''',0}}^{2j-\delta_{\rho''',-1}} \sum_{J_1=\delta_{\rho_1,0}}^{2j-\delta_{\rho_1,-1}} \sum_{m'=-Q'}^{Q'} \sum_{m'''=-Q'''}^{Q'''} \sum_{m_1=-Q_1}^{Q_1} \\
& \times (-1)^{m-1+m''+m_1} \hat{\mathcal{D}}_{J_1m_1(jj)\rho_1J'''m'''(jj)\rho'''}^{J-m(jj)} \hat{\mathcal{D}}_{J'm'(jj)\rho'J_1-m_1(jj)\rho_1}^{J''-m''(jj)} \\
& \times \text{tr}_k [2a_{\mu,Jm}(p) a_{\mu,J''m''}(r) y_{J'm',\rho'}(q) y_{J'''m''',\rho'''}(\ell) \\
& + 2x_{i,Jm}(p) x_{i,J''m''}(r) y_{J'm',\rho'}(q) y_{J'''m''',\rho'''}(\ell)]. \tag{3.39}
\end{aligned}$$

4 Loop corrections

The action obtained in the previous section is that of a 4D theory on $\mathbb{R}^2 \times (\text{fuzzy } S^2)$. We consider the following successive limits:

- **Step 1** (Moyal limit): Take $n = 2j + 1 \rightarrow \infty$ with the fuzziness $\Theta \equiv \frac{18}{M^2 n}$ and k fixed. Then, the IR and UV cutoffs on S^2 become $M \propto n^{-1/2} \rightarrow 0$ and $\Lambda_j \equiv \frac{M}{3} \cdot 2j \propto n^{1/2} \rightarrow \infty$, respectively. Namely, the fuzzy S^2 is decompactified to the noncommutative (Moyal) plane \mathbb{R}_Θ^2 .
- **Step 2** (commutative limit): Send the noncommutativity parameter Θ to zero.

At the level of the classical action or tree level amplitudes at least, the theory coincides with the ordinary $\mathcal{N} = 4$ $U(k)$ SYM on \mathbb{R}^4 after these steps. One might further expect that Step 1 would be safe even quantum mechanically since $4D$ $\mathcal{N} = 4$ SYM is UV finite and the deformation by the mass M would be soft. However, the situation is not so simple. Although the deformation parameter M giving masses naively seems soft, the softness is not clear in the sense that M gives not only the IR cutoff but also the UV cutoff. In addition, the obtained theory at Step 1 is a noncommutative field theory; i.e., there might appear nontrivial divergence through the so-called UV/IR mixing in non-planar diagrams [55]. In fact, we have to consider non-planar contributions as well as planar ones and take care of both the UV and IR divergences. In the following, we first compute the one-point functions at the one-loop order in the next section (section 5), and then explicitly calculate the two-point functions of the scalar fields X_i in section 6. This will give a check of whether we can take the limits safely even in the quantum mechanical sense.

Upon the perturbative calculation, we rescale all the fields as

$$(\text{field}) \rightarrow \frac{g_{2d}}{\sqrt{2n}} (\text{field}) \quad (4.1)$$

so that the kinetic terms take the canonical form. The kinetic terms of gauge fields a_μ and a scalar x_7 are written as

$$\int \frac{d^2 q}{(2\pi)^2} \sum_{J=0}^{2j} \sum_{m=-J}^J \frac{1}{2} (-1)^m \text{tr}_k \left[\varphi_{K,J-m}(-q) \tilde{\Delta}_J(q)_{KK'} \varphi_{K',Jm}(q) \right], \quad (4.2)$$

where $K, K' = 1, 2, 3$,

$$\varphi_{K=1,Jm} \equiv a_{1,Jm}, \quad \varphi_{K=2,Jm} \equiv a_{2,Jm}, \quad \varphi_{K=3,Jm} \equiv x_{7,Jm}, \quad (4.3)$$

and the kinetic kernel is a 3×3 matrix:

$$\tilde{\Delta}_J(q) \equiv \begin{bmatrix} q^2 + \frac{M^2}{9} J(J+1) & 0 & \frac{2M}{9} q_2 \\ 0 & q^2 + \frac{M^2}{9} J(J+1) & -\frac{2M}{9} q_1 \\ -\frac{2M}{9} q_2 & \frac{2M}{9} q_1 & q^2 + \frac{2M^2}{81} + \frac{M^2}{9} J(J+1) \end{bmatrix}. \quad (4.4)$$

Then, we obtain the propagator

$$\begin{aligned} & \langle \varphi_{K, Jm}(q)_{st} \varphi_{K', J'm'}(q')_{s't'} \rangle_0 \\ &= \delta_{s't'} \delta_{t s'} \delta_{J J'} \delta_{m+m', 0} (-1)^{m'} (2\pi)^2 \delta^2(q+q') \left(\tilde{\Delta}_J(q)^{-1} \right)_{KK'}. \end{aligned} \quad (4.5)$$

Here, $s, t, s', t' (= 1, \dots, k)$ denote the color indices. The inverse of the kinetic kernel is given by

$$\begin{aligned} \tilde{\Delta}_J(q)^{-1} &= \frac{1}{q^2 + \frac{M^2}{9} J(J+1)} \mathbb{1}_3 \\ &+ \frac{\frac{2M}{9}}{[q^2 + \frac{M^2}{9} J(J + \frac{1}{3})][q^2 + \frac{M^2}{9} (J+1)(J + \frac{2}{3})]} \begin{bmatrix} & -q_2 \\ & q_1 \\ q_2 & -q_1 & -\frac{M}{9} \end{bmatrix} \\ &+ \frac{\frac{4M^2}{81}}{[q^2 + \frac{M^2}{9} J(J+1)][q^2 + \frac{M^2}{9} J(J + \frac{1}{3})][q^2 + \frac{M^2}{9} (J+1)(J + \frac{2}{3})]} \begin{bmatrix} -q_2^2 & q_1 q_2 & \\ q_1 q_2 & -q_1^2 & \\ & & -q^2 \end{bmatrix}. \end{aligned} \quad (4.6)$$

The propagators for the other bosons and ghosts can be easily read off. For scalars,

$$\begin{aligned} & \langle x_{i, Jm}(q)_{st} x_{i', J'm'}(q')_{s't'} \rangle_0 \\ &= \delta_{i i'} \delta_{s t'} \delta_{t s'} \delta_{J J'} \delta_{m+m', 0} (-1)^{m'} (2\pi)^2 \delta^2(q+q') \frac{1}{q^2 + \frac{M^2}{9} (J + \frac{1}{3})(J + \frac{2}{3})} \end{aligned} \quad (4.7)$$

with $i, i' = 3, 4, 5, 6$. For ghosts and y -fields,

$$\begin{aligned} & \langle c_{Jm}(q)_{st} \bar{c}_{J'm'}(q')_{s't'} \rangle_0 \\ &= \delta_{s t'} \delta_{t s'} \delta_{J J'} \delta_{m+m', 0} (-1)^{m'} (2\pi)^2 \delta^2(q+q') \frac{-1}{q^2 + \frac{M^2}{9} J(J+1)}, \end{aligned} \quad (4.8)$$

$$\begin{aligned} & \langle y_{Jm, \rho}(q)_{st} y_{J'm', \rho'}(q')_{s't'} \rangle_0 \\ &= \delta_{\rho \rho'} \delta_{s t'} \delta_{t s'} \delta_{J J'} \delta_{m+m', 0} (-1)^{m'+1} (2\pi)^2 \delta^2(q+q') \tilde{\Delta}_{(\rho, J)}(q)^{-1} \end{aligned} \quad (4.9)$$

with

$$\tilde{\Delta}_{(\rho, J)}(q)^{-1} \equiv \frac{\delta_{\rho, 0}}{q^2 + \frac{M^2}{9} J(J+1)} + \frac{\delta_{\rho, 1} + \delta_{\rho, -1}}{q^2 + \frac{M^2}{9} (J+1)^2}. \quad (4.10)$$

Finally, the fermion propagators are obtained as

$$\left\langle \hat{\Psi}_{Jm, \kappa}(q)_{r, st} \hat{\Psi}_{J'm', \kappa'}(q')_{r', s't'} \right\rangle_0$$

$$= \delta_{\kappa, \kappa'} \delta_{s t'} \delta_{t s'} \delta_{J J'} \delta_{m+m', 0} (-1)^{m+\frac{1}{2}\kappa} (2\pi)^2 \delta^2(q+q') \left(\hat{D}_{J, \kappa}(q)^{-1} \right)_{r r'}, (4.11)$$

where $r, r' (= 1, \dots, 8)$ label the spinor components of (3.27), and the inverse of the kernel is given by

$$\begin{aligned} \hat{D}_{J, \kappa=1}(q)^{-1} &= \frac{1}{q^2 + \frac{M^2}{9}(J+1)^2} \\ &\quad \times \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & -\frac{M}{3}(J+1) & & & & & q_2 \\ & & & -\frac{M}{3}(J+1) & & & & q_1 \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & -\frac{M}{3}(J+1) \end{bmatrix} \\ &+ \frac{1}{q^2 + \frac{M^2}{9}(J+1)(J+\frac{2}{3})} \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & & & & & q_1 \\ & & & 0 & & & & -q_2 \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & -\frac{M}{3}(J+1) \\ & & & & & & & & 0 \end{bmatrix} \\ &+ \frac{1}{q^2 + \frac{M^2}{9}(J+\frac{1}{3})(J+\frac{2}{3})} \\ &\quad \times \begin{bmatrix} -\frac{M}{3}(J+\frac{2}{3}) & & & & q_1 & & -q_2 \\ & -\frac{M}{3}(J+\frac{2}{3}) & & & q_2 & & q_1 \\ & & 0 & & & & \\ & & & 0 & & & \\ -q_1 & & -q_2 & & -\frac{M}{3}(J+\frac{1}{3}) & & \\ q_2 & & -q_1 & & & -\frac{M}{3}(J+\frac{1}{3}) & \\ & & & & & & 0 \\ & & & & & & & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& + \frac{\frac{M}{9}}{[q^2 + \frac{M^2}{9}(J+1)^2][q^2 + \frac{M^2}{9}(J+1)(J+\frac{2}{3})]} \\
& \times \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & q_1^2 & -q_1 q_2 & & & & \\ & & -q_1 q_2 & q_2^2 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix} \\
& \quad (4.12)
\end{aligned}$$

and

$$\begin{aligned}
\hat{D}_{J,\kappa=-1}(q)^{-1} &= \frac{1}{q^2 + \frac{M^2}{9}(J+\frac{1}{2})^2} \\
& \times \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & \frac{M}{3}(J+\frac{1}{2}) & & & & q_2 & \\ & & & \frac{M}{3}(J+\frac{1}{2}) & & & q_1 & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & -q_2 & -q_1 & & & & \frac{M}{3}(J+\frac{1}{2}) \end{bmatrix} \\
& + \frac{1}{q^2 + \frac{M^2}{9}(J+\frac{1}{2})(J+\frac{5}{6})} \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & & q_1 & & & \\ & & & 0 & -q_2 & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & -q_1 & q_2 & & \frac{M}{3}(J+\frac{1}{2}) & & & \\ & & & & & & 0 & \end{bmatrix} \\
& + \frac{1}{q^2 + \frac{M^2}{9}(J+\frac{5}{6})(J+\frac{7}{6})}
\end{aligned}$$

$$\begin{aligned}
& \times \begin{bmatrix} \frac{M}{3}(J + \frac{5}{6}) & & & q_1 & -q_2 \\ & \frac{M}{3}(J + \frac{5}{6}) & & q_2 & q_1 \\ & & 0 & & \\ & & & 0 & \\ -q_1 & -q_2 & & \frac{M}{3}(J + \frac{7}{6}) & \\ q_2 & -q_1 & & \frac{M}{3}(J + \frac{7}{6}) & \\ & & & & 0 \\ & & & & 0 \end{bmatrix} \\
& + \frac{\frac{M}{9}}{[q^2 + \frac{M^2}{9}(J + \frac{1}{2})^2][q^2 + \frac{M^2}{9}(J + \frac{1}{2})(J + \frac{5}{6})]} \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & q_1^2 & -q_1 q_2 & & \\ & & -q_1 q_2 & q_2^2 & & \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \end{bmatrix}.
\end{aligned} \tag{4.13}$$

5 One-point functions at one-loop

Due to the Q_{\pm} supersymmetries, gauge invariant one-point functions should not be induced radiatively for any $n = 2j + 1$ and M . As a warm-up exercise, let us check this at the one-loop level by computing tadpole diagrams. We also see that the one-point functions do not contribute to the one-loop effective action in the Moyal limit. In this section, \bar{p} and (\bar{J}, \bar{m}) denote external momentum and angular momentum, respectively.

5.1 One-point functions of x_i and a_{μ}

The one-loop contribution to $\langle \frac{1}{k} \text{tr}_k x_{i, \bar{J} \bar{m}}(\bar{p}) \rangle$ ($i = 3, 4, 5, 6$) comes from the Wick contractions among $\frac{1}{k} \text{tr}_k x_{i, \bar{J} \bar{m}}(\bar{p})$ and $-S_{3,F}^{\mathcal{F}}$. We obtain

$$\begin{aligned}
\left\langle \frac{1}{k} \text{tr}_k x_{i, \bar{J} \bar{m}}(\bar{p}) \right\rangle &= \frac{g_{2d}}{\sqrt{2n}} k \delta^2(\bar{p}) (-1)^{-\bar{m}} \int^{\Lambda_p} d^2 p \sum_{\kappa=\pm 1} \sum_{J=\frac{1}{2}\delta_{\kappa,-1}}^{2j-\frac{1}{2}\delta_{\kappa,-1}} \sum_{m=-U}^U \\
&\times \hat{\mathcal{F}}_{J-m(jj)_{\kappa} \bar{J}-\bar{m}(jj)}^{J-m(jj)_{\kappa}} \frac{1}{\frac{M^2}{9}(\bar{J} + \frac{1}{3})(\bar{J} + \frac{2}{3})} \text{tr}_8 \left(\hat{\gamma}_i \hat{D}_{J,\kappa}(p)^{-1} \right) \tag{5.1}
\end{aligned}$$

at the one-loop order. We introduced a cutoff Λ_p for the loop momentum integration. “tr₈” stands for trace over spinor indices. It is easy to see that $\text{tr}_8 \left(\hat{\gamma}_i \hat{D}_{J,\kappa}(p)^{-1} \right)$ vanishes for each i and κ . Hence,

$$\left\langle \frac{1}{k} \text{tr}_k x_{i, \bar{J} \bar{m}}(\bar{p}) \right\rangle = 0 \quad \text{at the one-loop order.} \quad (5.2)$$

The one-loop contribution to $\left\langle \frac{1}{k} \text{tr}_k x_{7, \bar{J} \bar{m}}(\bar{p}) \right\rangle$ can be written as

$$\left\langle \frac{1}{k} \text{tr}_k x_{7, \bar{J} \bar{m}}(\bar{p}) \right\rangle = \left\langle \frac{1}{k} \text{tr}_k x_{7, \bar{J} \bar{m}}(\bar{p}) (-S_{3,B}^C - S_{3,B}^D - S_{3,F}^F) \right\rangle_{1\text{-loop}}, \quad (5.3)$$

where the subscript “1-loop” means the Wick contractions generating one-loop diagrams. We see that each of the three contributions arising from the contractions with $-S_{3,B}^C$, $-S_{3,B}^D$ and $-S_{3,F}^F$ vanishes separately, because the integrands are odd functions of loop momenta or vanish by themselves due to

$$p_\mu \left(\tilde{\Delta}_J(p)^{-1} \right)_{\mu 3} = p_\mu \left(\tilde{\Delta}_J(p)^{-1} \right)_{3 \mu} = 0, \quad \text{tr}_8 \left(\hat{\gamma}_7 \hat{D}_{J,\kappa}(p)^{-1} \right) = 0, \quad (5.4)$$

which leads to

$$\left\langle \frac{1}{k} \text{tr}_k x_{7, \bar{J} \bar{m}}(\bar{p}) \right\rangle = 0 \quad \text{at the one-loop order.} \quad (5.5)$$

$\left\langle \frac{1}{k} \text{tr}_k a_{\mu, \bar{J} \bar{m}}(\bar{p}) \right\rangle$ is similar to the situation for x_7 :

$$\left\langle \frac{1}{k} \text{tr}_k a_{\mu, \bar{J} \bar{m}}(\bar{p}) \right\rangle = \left\langle \frac{1}{k} \text{tr}_k a_{\mu, \bar{J} \bar{m}}(\bar{p}) (-S_{3,B}^C - S_{3,B}^D - S_{3,F}^F) \right\rangle_{1\text{-loop}}, \quad (5.6)$$

and all the loop integrals vanish for the same reason as in x_7 ;

$$\left\langle \frac{1}{k} \text{tr}_k a_{\mu, \bar{J} \bar{m}}(\bar{p}) \right\rangle = 0 \quad \text{at the one-loop order.} \quad (5.7)$$

5.2 One-point function of y

The one-point function of y can be written as

$$\left\langle \frac{1}{k} \text{tr}_k y_{\bar{J} \bar{m}, \bar{\rho}}(\bar{p}) \right\rangle = \left\langle \frac{1}{k} \text{tr}_k y_{\bar{J} \bar{m}, \bar{\rho}}(\bar{p}) (-S_{3,B}^E - S_{3,B}^D - S_{3,F}^G) \right\rangle_{1\text{-loop}}. \quad (5.8)$$

The contractions with $-S_{3,B}^E$ and $-S_{3,B}^D$ lead to diagrams with loops of bosons or ghosts, and the contractions with $-S_{3,F}^G$ to loop diagrams of fermions. This time each diagram does not vanish separately. Let us compute these three contributions explicitly.

First contribution The first contribution is tadpoles of y -loops:

$$\begin{aligned}
& \left\langle \frac{1}{k} \text{tr}_k y_{\bar{J}\bar{m},\bar{\rho}}(\bar{p}) (-S_{3,B}^{\mathcal{E}}) \right\rangle_{1\text{-loop}} \\
&= -\frac{g_{2d}}{\sqrt{2n}} k i \frac{M}{3} \delta^2(\bar{p}) (-1)^{-\bar{m}} \tilde{\Delta}_{(\bar{\rho},\bar{J})}(\bar{p}) \sum_{\rho'=-1}^1 \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} [\bar{\rho}(\bar{J}+1) + 2\rho'(J'+1)] \\
&\quad \times \left(\sum_{m'=-Q'}^{Q'} (-1)^{-m'} \hat{\mathcal{E}}_{\bar{J}-\bar{m}(jj) \bar{\rho} J' m'(jj) \rho' J'-m'(jj) \rho'} \right) \int^{\Lambda_p} d^2p \tilde{\Delta}_{(\rho',J')}(p). \quad (5.9)
\end{aligned}$$

The sum of $\hat{\mathcal{E}}$ is calculated in appendix C.1. Plugging the result (C.7) into (5.9) leads to

$$\begin{aligned}
& \left\langle \frac{1}{k} \text{tr}_k y_{\bar{J}\bar{m},\bar{\rho}}(\bar{p}) (-S_{3,B}^{\mathcal{E}}) \right\rangle_{1\text{-loop}} = \frac{g_{2d}}{\sqrt{2n}} i \frac{3k}{2M} \frac{1}{\sqrt{j(j+1)}} \delta^2(\bar{p}) \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0} \\
& \times \sum_{J'=1}^{2j} \int^{\Lambda_p} d^2p \left\{ \frac{2J'+1}{p^2 + \frac{M^2}{9} J'(J'+1)} + \frac{J'(2J'+1)(2J'+3)}{p^2 + \frac{M^2}{9} (J'+1)^2} + \frac{(J'+1)(2J'-1)(2J'+1)}{p^2 + \frac{M^2}{9} J'^2} \right\}. \quad (5.10)
\end{aligned}$$

Second contribution The second contribution is tadpoles of boson (a_μ, x_i) loops and ghost loops:

$$\begin{aligned}
& \left\langle \frac{1}{k} \text{tr}_k y_{\bar{J}\bar{m},\bar{\rho}}(\bar{p}) (-S_{3,B}^{\mathcal{D}}) \right\rangle_{1\text{-loop}} = \frac{g_{2d}}{\sqrt{2n}} \frac{kM}{3} \delta^2(\bar{p}) (-1)^{-\bar{m}} \tilde{\Delta}_{(\bar{\rho},\bar{J})}(\bar{p}) \\
& \times \sum_{J=1}^{2j} \sqrt{J(J+1)} \left\{ \sum_{m=-J}^J \left(\hat{\mathcal{D}}_{\bar{J}-\bar{m}(jj) \bar{\rho} J m(jj) 0}^{J m(jj)} - \hat{\mathcal{D}}_{J m(jj) 0 \bar{J}-\bar{m}(jj) \bar{\rho}}^{J m(jj)} \right) \right\} \\
& \times \int^{\Lambda_p} d^2p \left\{ \left(\tilde{\Delta}_J(p)^{-1} \right)_{KK} + \frac{4}{p^2 + \frac{M^2}{9} (J + \frac{1}{3})(J + \frac{2}{3})} + \frac{-1}{p^2 + \frac{M^2}{9} J(J+1)} \right\}. \quad (5.11)
\end{aligned}$$

From the result of the sum of $\hat{\mathcal{D}}$, (C.18), we find

$$\begin{aligned}
& \left\langle \frac{1}{k} \text{tr}_k y_{\bar{J}\bar{m},\bar{\rho}}(\bar{p}) (-S_{3,B}^{\mathcal{D}}) \right\rangle_{1\text{-loop}} \\
&= \frac{g_{2d}}{\sqrt{2n}} i \frac{3k}{M} \frac{1}{\sqrt{j(j+1)}} \delta^2(\bar{p}) \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0} \sum_{J=1}^{2j} J(J+1)(2J+1) \\
& \times \int^{\Lambda_p} d^2p \left\{ \frac{2}{p^2 + \frac{M^2}{9} J(J+1)} + \frac{4}{p^2 + \frac{M^2}{9} (J + \frac{1}{3})(J + \frac{2}{3})} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{-\frac{2M^2}{81}}{[p^2 + \frac{M^2}{9}J(J + \frac{1}{3})][p^2 + \frac{M^2}{9}(J+1)(J + \frac{2}{3})]} \\
& + \frac{-\frac{8M^2}{81}p^2}{[p^2 + \frac{M^2}{9}J(J+1)][p^2 + \frac{M^2}{9}J(J + \frac{1}{3})][p^2 + \frac{M^2}{9}(J+1)(J + \frac{2}{3})]} \Bigg\}.
\end{aligned} \tag{5.12}$$

Third contribution The third contribution is tadpoles of fermion loops:

$$\begin{aligned}
\left\langle \frac{1}{k} \text{tr}_k y_{\bar{J}\bar{m},\bar{\rho}}(\bar{p}) (-S_{3,F}^{\mathcal{G}}) \right\rangle_{1\text{-loop}} &= \frac{g_{2d}}{\sqrt{2n}} k \delta^2(\bar{p}) (-1)^{-\bar{m}+1} \tilde{\Delta}_{(\bar{\rho},\bar{J})}(\bar{p}) \\
&\times \sum_{\kappa=\pm 1} \sum_{J=\frac{1}{2}\delta_{\kappa,-1}}^{2j-\frac{1}{2}\delta_{\kappa,-1}} \left(\sum_{m=-U}^U \hat{\mathcal{G}}_{J-m(jj)\kappa\bar{J}-\bar{m}(jj)\bar{\rho}}^{J-m(jj)\kappa} \right) \int^{\Lambda_p} d^2p \text{tr}_8 \left(\hat{D}_{J,\kappa}(p)^{-1} \right). \tag{5.13}
\end{aligned}$$

Here, the spinor trace reads

$$\begin{aligned}
\text{tr}_8 \left(\hat{D}_{J,\kappa=1}(p)^{-1} \right) &= \frac{-M(J+1)}{p^2 + \frac{M^2}{9}(J+1)^2} + \frac{-\frac{M}{3}(J+1)}{p^2 + \frac{M^2}{9}(J+1)(J + \frac{2}{3})} \\
&+ \frac{-\frac{M}{3}(4J+2)}{p^2 + \frac{M^2}{9}(J + \frac{1}{3})(J + \frac{2}{3})} + \frac{\frac{M}{9}p^2}{[p^2 + \frac{M^2}{9}(J+1)^2][p^2 + \frac{M^2}{9}(J+1)(J + \frac{2}{3})]} \tag{5.14}
\end{aligned}$$

and

$$\begin{aligned}
\text{tr}_8 \left(\hat{D}_{J,\kappa=-1}(p)^{-1} \right) &= \frac{M(J + \frac{1}{2})}{p^2 + \frac{M^2}{9}(J + \frac{1}{2})^2} + \frac{\frac{M}{3}(J + \frac{1}{2})}{p^2 + \frac{M^2}{9}(J + \frac{1}{2})(J + \frac{5}{6})} \\
&+ \frac{\frac{M}{3}(4J+4)}{p^2 + \frac{M^2}{9}(J + \frac{5}{6})(J + \frac{7}{6})} + \frac{\frac{M}{9}p^2}{[p^2 + \frac{M^2}{9}(J + \frac{1}{2})^2][p^2 + \frac{M^2}{9}(J + \frac{1}{2})(J + \frac{5}{6})]}. \tag{5.15}
\end{aligned}$$

The sum of $\hat{\mathcal{G}}$ is computed in (C.23). Plugging these into (5.13), we obtain

$$\begin{aligned}
\left\langle \frac{1}{k} \text{tr}_k y_{\bar{J}\bar{m},\bar{\rho}}(\bar{p}) (-S_{3,F}^{\mathcal{G}}) \right\rangle_{1\text{-loop}} &= \frac{g_{2d}}{\sqrt{2n}} i \frac{9k}{M} \frac{1}{\sqrt{j(j+1)}} \delta^2(\bar{p}) \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0} \sum_{J=1}^{2j} J(J+1) \\
&\times \int^{\Lambda_p} d^2p \left\{ \frac{-(J+1)}{p^2 + \frac{M^2}{9}(J+1)^2} + \frac{-\frac{1}{3}(J+1)}{p^2 + \frac{M^2}{9}(J+1)(J + \frac{2}{3})} \right. \\
&+ \frac{-J}{p^2 + \frac{M^2}{9}J^2} + \frac{-\frac{1}{3}J}{p^2 + \frac{M^2}{9}J(J + \frac{1}{3})} + \frac{-\frac{8}{3}(J + \frac{1}{2})}{p^2 + \frac{M^2}{9}(J + \frac{1}{3})(J + \frac{2}{3})} \\
&+ \left. \frac{\frac{1}{9}p^2}{[p^2 + \frac{M^2}{9}(J+1)^2][p^2 + \frac{M^2}{9}(J+1)(J + \frac{2}{3})]} + \frac{-\frac{1}{9}p^2}{[p^2 + \frac{M^2}{9}J^2][p^2 + \frac{M^2}{9}J(J + \frac{1}{3})]} \right\}. \tag{5.16}
\end{aligned}$$

Total contribution Gathering the three contributions (5.10), (5.12), and (5.16), the one-point function becomes

$$\begin{aligned} \left\langle \frac{1}{k} \text{tr}_k y_{\bar{J}\bar{m},\bar{\rho}}(\bar{p}) \right\rangle &= \frac{g_{2d}}{\sqrt{2n}} i \frac{3k}{2M} \frac{1}{\sqrt{j(j+1)}} \delta^2(\bar{p}) \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0} \\ &\times \sum_{J=1}^{2j} \int^{\Lambda_p} d^2p \left\{ \frac{2J+1}{p^2 + \frac{M^2}{9}J(J+1)} + \frac{-(J+1)}{p^2 + \frac{M^2}{9}J^2} + \frac{-J}{p^2 + \frac{M^2}{9}(J+1)^2} \right\}. \end{aligned} \quad (5.17)$$

Here, we see that the $\mathcal{O}(1/p^2)$ and $\mathcal{O}(1/p^4)$ terms of the large- $|p|$ expansion of the integrand vanish and the integral converges owing to the Q_{\pm} supersymmetries. In fact, after the p -integrals and the summation of J , we end up with

$$\left\langle \frac{1}{k} \text{tr}_k y_{\bar{J}\bar{m},\bar{\rho}}(\bar{p}) \right\rangle = \frac{g_{2d}}{\sqrt{2n}} i \frac{3k}{M} \frac{-\pi \ln n}{\sqrt{n^2 - 1}} \delta^2(\bar{p}) \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0}. \quad (5.18)$$

This indicates that $\text{tr}_k y_{00,\rho=-1}$ is generated by a one-loop effect. However, we should note that $\text{tr}_k y_{00,\rho=-1}$ is not gauge invariant, as discussed in appendix C.4.

Let us consider the gauge invariant combination from (C.25):

$$\text{Tr } \vec{Y}(p) = \sum_{\rho=-1}^1 \sum_{J=\delta_{\rho,0}}^{2j-\delta_{\rho,-1}} \sum_{m=-Q}^Q \left(\text{tr}_n \vec{Y}_{Jm(jj)}^{\rho} \right) \left(\text{tr}_k y_{Jm,\rho}(p) \right). \quad (5.19)$$

Its expectation value at the one-loop level becomes

$$\left\langle \text{Tr } \vec{Y}(p) \right\rangle = \left(\text{tr}_n \vec{Y}_{00(jj)}^{\rho=-1} \right) \left\langle \text{tr}_k y_{Jm,\rho}(p) \right\rangle, \quad (5.20)$$

since $\langle \text{tr}_k y_{Jm,\rho}(p) \rangle \propto \delta_{\rho,-1} \delta_{J0} \delta_{m0}$ from (5.18). It is shown in appendix C.5 that

$$\text{tr}_n \vec{Y}_{00(jj)}^{\rho=-1} = 0. \quad (5.21)$$

Thus, we conclude that the expectation value of the gauge invariant part $\left\langle \text{Tr } \vec{Y}(p) \right\rangle$ vanishes at the one-loop level.

5.3 Summary of the one-point functions

The results obtained in this section give an explicit check at the one-loop order for the statement that any gauge invariant one-point operators are not radiatively induced for arbitrary n and M .

Equation (5.18) multiplied by $M^2/9$, which is the one-point function with the external line truncated, is seen to vanish in the Moyal limit (Step 1 in the successive limits). Therefore, the one-point functions give no contribution in the one-loop effective action in the successive limits.

6 Two-point functions of scalar fields at one-loop

We express the two-point functions of scalar fields X_i as

$$-\frac{g_{2d}^2}{2n} \left(\frac{3}{M} \right)^2 \int \frac{d^2 p}{(2\pi)^2} \sum_{(J,m)} \sum_{(\bar{J},\bar{m})} (-1)^m \left\{ k \operatorname{tr}_k (x_{i,Jm}(p) x_{i,\bar{J}\bar{m}}(-p)) \mathcal{A}_{i,\bar{i}}(p; Jm; \bar{J}\bar{m}) \right. \\ \left. + \operatorname{tr}_k (x_{i,Jm}(p)) \operatorname{tr}_k (x_{i,\bar{J}\bar{m}}(-p)) \mathcal{B}_{i,\bar{i}}(p; Jm; \bar{J}\bar{m}) \right\}. \quad (6.1)$$

Here, $\mathcal{A}_{i,\bar{i}}(p; Jm; \bar{J}\bar{m})$ and $\mathcal{B}_{i,\bar{i}}(p; Jm; \bar{J}\bar{m})$ are contributions from planar and non-planar diagrams, respectively, whose external legs are removed.

6.1 List of one-loop graphs

Let us write each of $\mathcal{A}_{i,\bar{i}}$ and $\mathcal{B}_{i,\bar{i}}$ as a summation of six contributions:

$$\mathcal{A}_{i,\bar{i}} = \mathcal{A}_{i,\bar{i}}^{4CC} + \mathcal{A}_{i,\bar{i}}^{4CD} + \mathcal{A}_{i,\bar{i}}^{3DD} + \mathcal{A}_{i,\bar{i}}^{3FF} + \mathcal{A}_{i,\bar{i}}^{3CC(1)} + \mathcal{A}_{i,\bar{i}}^{3CC(2)}, \\ \mathcal{B}_{i,\bar{i}} = \mathcal{B}_{i,\bar{i}}^{4CC} + \mathcal{B}_{i,\bar{i}}^{4CD} + \mathcal{B}_{i,\bar{i}}^{3DD} + \mathcal{B}_{i,\bar{i}}^{3FF} + \mathcal{B}_{i,\bar{i}}^{3CC(1)} + \mathcal{B}_{i,\bar{i}}^{3CC(2)}. \quad (6.2)$$

The superscript $4XY$ means contributions from diagrams containing a single 4-point vertex that has the vertex coefficients \hat{X} and \hat{Y} , while $3XY$ stands for contributions from diagrams consisting of two 3-point vertices one of which includes the vertex coefficient \hat{X} and the other has \hat{Y} ($X, Y = C, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}$). In (6.2), we have further divided each of $\mathcal{A}_{i,\bar{i}}^{3CC}$ and $\mathcal{B}_{i,\bar{i}}^{3CC}$ into two parts with the symbols “(1)” and “(2)”, in which the latter reflects that the interaction terms $S_{3,B}^C$ including $x_{3,Jm}(p)$, $x_{4,Jm}(p)$, and $x_{7,Jm}(p)$ yield extra contributions to $\mathcal{A}_{3,3}^{3CC}$, $\mathcal{A}_{4,4}^{3CC}$, $\mathcal{A}_{3,4}^{3CC}$, $\mathcal{A}_{7,7}^{3CC}$, $\mathcal{B}_{3,3}^{3CC}$, $\mathcal{B}_{4,4}^{3CC}$, $\mathcal{B}_{3,4}^{3CC}$, and $\mathcal{B}_{7,7}^{3CC}$. In addition, due to the propagator connecting x_7 and a_μ (4.5), the expressions of $\mathcal{A}_{i,\bar{i}}$ and $\mathcal{B}_{i,\bar{i}}$ are different depending on whether the external line includes $x_{7,Jm}(p)$ or not. So we separately treat $\mathcal{A}_{i,\bar{i}}$, $\mathcal{B}_{i,\bar{i}}$ with $(i, \bar{i}) \in \{3, \dots, 6\}$ and $\mathcal{A}_{7,7}$, $\mathcal{B}_{7,7}$. Note that $\mathcal{A}_{i,7}$ and $\mathcal{B}_{i,7}$ are identically zero.

In the following, we explicitly present all the contributions that do not trivially vanish. In order to simplify notations, the following symbols are employed:

$$\sum_{(J,m)} \equiv \sum_{J=0}^{2j} \sum_{m=-J}^J, \quad \sum_{(\rho,J,m)} \equiv \sum_{\rho=-1}^1 \sum_{J=\delta_{\rho,0}}^{2j-\delta_{\rho,-1}} \sum_{m=-Q}^Q, \quad \sum_{[\kappa,J,m]} \equiv \sum_{\kappa=\pm 1} \sum_{J=\frac{1}{2}\delta_{\kappa,-1}}^{2j-\frac{1}{2}\delta_{\kappa,-1}} \sum_{m=-U}^U \quad (6.3)$$

with $Q = J + \delta_{\rho,1}$ and $U = J + \frac{1}{2}\delta_{\kappa,1}$. In expressing index structures of planar and non-planar contributions, the relations (B.24), (B.99), and (B.118) are used.

6.1.1 $\mathcal{A}_{i,\bar{i}}$ and $\mathcal{B}_{i,\bar{i}}$ with $i, \bar{i} = 3, \dots, 6$

The planar contributions are ³

$$\begin{aligned} \mathcal{A}_{i,i}^{4CC} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{m''-m'} \hat{C}_{Jm(jj) \bar{J}\bar{m}(jj)}^{J''m''(jj)} \hat{C}_{J'm'(jj) J'-m'(jj)}^{J''-m''(jj)} \\ & \times \int \frac{d^2 q}{(2\pi)^2} \left\{ \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{KK} + \frac{3}{q^2 + \frac{M^2}{9}(J' + \frac{1}{3})(J' + \frac{2}{3})} \right\}, \end{aligned} \quad (6.4)$$

$$\begin{aligned} \mathcal{A}_{i,i}^{4CD} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(\rho',J',m')} \sum_{(J'',m'')} (-1)^{m''-m'+1} \hat{C}_{Jm(jj) \bar{J}\bar{m}(jj)}^{J''m''(jj)} \hat{\mathcal{D}}_{J'm'(jj) \rho' J'-m'(jj) \rho'}^{J''-m''(jj)} \\ & \times \int \frac{d^2 q}{(2\pi)^2} \tilde{\Delta}_{(\rho',J')}(q)^{-1}, \end{aligned} \quad (6.5)$$

$$\begin{aligned} \mathcal{A}_{i,i}^{3DD} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(J',m')} \sum_{(\rho'',J'',m'')} (-1)^{-m'-m''} \\ & \times \left((-1)^{m'} \sqrt{J(J+1)} \hat{\mathcal{D}}_{J''m''(jj) \rho'' Jm(jj) 0}^{J'-m'(jj)} - (-1)^m \sqrt{J'(J'+1)} \hat{\mathcal{D}}_{J'm'(jj) 0 J''m''(jj) \rho''}^{J-m(jj)} \right) \\ & \times \left((-1)^{\bar{m}} \sqrt{J'(\bar{J}+1)} \hat{\mathcal{D}}_{J''-m''(jj) \rho'' J'-m'(jj) 0}^{\bar{J}-\bar{m}(jj)} - (-1)^{-m'} \sqrt{\bar{J}(\bar{J}+1)} \hat{\mathcal{D}}_{\bar{J}\bar{m}(jj) 0 J''-m''(jj) \rho''}^{J'-m'(jj)} \right) \\ & \times \frac{M^2}{9} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 + \frac{M^2}{9}(J' + \frac{1}{3})(J' + \frac{2}{3})} \tilde{\Delta}_{(\rho'',J'')}(p+q)^{-1}, \end{aligned} \quad (6.6)$$

$$\begin{aligned} \mathcal{A}_{i,i}^{3FF} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{[\kappa',J',m']} \sum_{[\kappa'',J'',m'']} \hat{\mathcal{F}}_{J''m''(jj) \kappa'' \bar{J}\bar{m}(jj)}^{J'm'(jj) \kappa'} \hat{\mathcal{F}}_{J'm'(jj) \kappa' Jm(jj)}^{J''m''(jj) \kappa''} \\ & \times \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \text{tr}_8 \left(\hat{\gamma}_i \hat{D}_{J',\kappa'}(q)^{-1} \hat{\gamma}_{\bar{i}} \hat{D}_{J'',\kappa''}(p+q)^{-1} \right), \end{aligned} \quad (6.7)$$

$$\begin{aligned} \mathcal{A}_{i,i}^{3CC(1)} = & (-1)^{m+1} \left(\frac{M}{3} \right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{m'-m''} \hat{C}_{Jm(jj) J''m''(jj)}^{J'm'(jj)} \hat{C}_{J''-m''(jj) \bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\ & \times \int \frac{d^2 q}{(2\pi)^2} (q_\mu + 2p_\mu)(q_\nu + 2p_\nu) \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu\nu} \frac{1}{(p+q)^2 + \frac{M^2}{9}(J'' + \frac{1}{3})(J'' + \frac{2}{3})}, \end{aligned} \quad (6.8)$$

$$\begin{aligned} \mathcal{A}_{3,3}^{3CC(2)} = & \mathcal{A}_{4,4}^{3CC(2)} = (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{m'-m''} \hat{C}_{Jm(jj) J''m''(jj)}^{J'm'(jj)} \hat{C}_{J''-m''(jj) \bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\ & \times \left(\frac{2M}{9} \right)^2 \int \frac{d^2 q}{(2\pi)^2} \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{33} \frac{1}{(p+q)^2 + \frac{M^2}{9}(J'' + \frac{1}{3})(J'' + \frac{2}{3})}, \end{aligned} \quad (6.9)$$

³Here, i and \bar{i} run from 3 to 6, excluding 7. In what follows, the sum over i is not assumed in $\mathcal{A}_{i,i}$ or $\mathcal{B}_{i,i}$.

$$\begin{aligned}
\mathcal{A}_{3,4}^{3CC(2)} = -\mathcal{A}_{4,3}^{3CC(2)} = (-1)^{m+1} \left(\frac{M}{3}\right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{m'-m''} \\
\times \hat{C}_{Jm(jj)J''m''(jj)}^{J'm'(jj)} \hat{C}_{J''-m''(jj)\bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\
\times \frac{4M}{9} \int \frac{d^2q}{(2\pi)^2} (q_\mu + 2p_\mu) \left(\tilde{\Delta}_{J'}(q)^{-1}\right)_{\mu 3} \frac{1}{(p+q)^2 + \frac{M^2}{9}(J'' + \frac{1}{3})(J'' + \frac{2}{3})}. \quad (6.10)
\end{aligned}$$

Corresponding to the second expression of S_4^{CD} (3.39), $\mathcal{A}_{i,i}^{4CD}$ can also be written as

$$\begin{aligned}
\mathcal{A}_{i,i}^{4CD} = (-1)^{\bar{m}} \left(\frac{M}{3}\right)^2 \sum_{(\rho',J',m')} \sum_{(\rho'',J'',m'')} (-1)^{m''-m'} \\
\times \hat{\mathcal{D}}_{J''m''(jj)\rho''J'-m'(jj)\rho'}^{J-m(jj)} \hat{\mathcal{D}}_{J'm'(jj)\rho'J''-m''(jj)\rho''}^{\bar{J}-\bar{m}(jj)} \int \frac{d^2q}{(2\pi)^2} \tilde{\Delta}_{(\rho',J')}(q)^{-1}. \quad (6.11)
\end{aligned}$$

We can see that off-diagonal parts of $\mathcal{A}_{i,\bar{i}}^{3FF}$ with $(i,\bar{i}) = (3,4)$ and $(5,6)$ are identically zero, and that those with $(i,\bar{i}) = (3,5)$, $(3,6)$, $(4,5)$ and $(4,6)$ are nonzero but become irrelevant in the Moyal limit, for instance, by using Theorem 1 and 2 in section 6.2.

The non-planar contributions are

$$\begin{aligned}
\mathcal{B}_{i,i}^{4CC} = \left(\frac{M}{3}\right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{J'+J''-J+1} (-1)^{m''-m'} \hat{C}_{J'm'(jj)Jm(jj)}^{J''m''(jj)} \hat{C}_{J''m''(jj)\bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\
\times \int \frac{d^2q}{(2\pi)^2} \left\{ \left(\tilde{\Delta}_{J'}(q)^{-1}\right)_{KK} + \frac{3}{q^2 + \frac{M^2}{9}(J' + \frac{1}{3})(J' + \frac{2}{3})} \right\}, \quad (6.12)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{i,i}^{4DD} = (-1)^{\bar{m}} \left(\frac{M}{3}\right)^2 \sum_{(\rho',J',m')} \sum_{(\rho'',J'',m'')} (-1)^{\bar{Q}'+\bar{Q}''-J+1} (-1)^{m''-m'} \\
\times \hat{\mathcal{D}}_{J''m''(jj)\rho''J'-m'(jj)\rho'}^{J-m(jj)} \hat{\mathcal{D}}_{J'm'(jj)\rho'J''-m''(jj)\rho''}^{\bar{J}-\bar{m}(jj)} \int \frac{d^2q}{(2\pi)^2} \tilde{\Delta}_{(\rho',J')}(q)^{-1}, \quad (6.13)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{i,i}^{3DD} = (-1)^m \left(\frac{M}{3}\right)^2 \sum_{(J',m')} \sum_{(\rho'',J'',m'')} (-1)^{J'+\bar{Q}''-J+1} (-1)^{-m'-m''} \\
\times \left((-1)^{m'} \sqrt{J(J+1)} \hat{\mathcal{D}}_{J''m''(jj)\rho''Jm(jj)0}^{J'-m'(jj)} - (-1)^m \sqrt{J'(J'+1)} \hat{\mathcal{D}}_{J'm'(jj)0J''m''(jj)\rho''}^{J-m(jj)} \right) \\
\times \left((-1)^{\bar{m}} \sqrt{J'(J'+1)} \hat{\mathcal{D}}_{J''-m''(jj)\rho''J'-m'(jj)0}^{\bar{J}-\bar{m}(jj)} - (-1)^{m'} \sqrt{\bar{J}(\bar{J}+1)} \hat{\mathcal{D}}_{\bar{J}\bar{m}(jj)0J''-m''(jj)\rho''}^{J'-m'(jj)} \right) \\
\times \frac{M^2}{9} \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + \frac{M^2}{9}(J' + \frac{1}{3})(J' + \frac{2}{3})} \tilde{\Delta}_{(\rho'',J'')}(p+q)^{-1}, \quad (6.14)
\end{aligned}$$

$$\mathcal{B}_{i,\bar{i}}^{3FF} = (-1)^m \left(\frac{M}{3}\right)^2 \sum_{[\kappa',J',m']} \sum_{[\kappa'',J'',m'']} (-1)^{\tilde{U}'+\tilde{U}''-J+1} \hat{\mathcal{F}}_{J'm'(jj)\kappa'Jm(jj)}^{J''m''(jj)\kappa''} \hat{\mathcal{F}}_{J''m''(jj)\kappa''\bar{J}\bar{m}(jj)}^{J'-m'(jj)\kappa'}$$

$$\times \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \text{tr}_8 \left(\hat{\gamma}_i \hat{D}_{J', \kappa'}(q)^{-1} \hat{\gamma}_{\bar{i}} \hat{D}_{J'', \kappa''}(p+q)^{-1} \right), \quad (6.15)$$

$$\begin{aligned} \mathcal{B}_{i,i}^{3CC(1)} = & (-1)^{m+1} \left(\frac{M}{3} \right)^2 \sum_{(J', m')} \sum_{(J'', m'')} (-1)^{J'+J''-J+1} (-1)^{m'-m''} \\ & \times \hat{C}_{Jm(jj)J''m''(jj)}^{J'm'(jj)} \hat{C}_{J''-m''(jj)\bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\ & \times \int \frac{d^2 q}{(2\pi)^2} (q_\mu + 2p_\mu)(q_\nu + 2p_\nu) \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu\nu} \frac{1}{(p+q)^2 + \frac{M^2}{9}(J'' + \frac{1}{3})(J'' + \frac{2}{3})}, \end{aligned} \quad (6.16)$$

$$\begin{aligned} \mathcal{B}_{3,3}^{3CC(2)} = & \mathcal{B}_{4,4}^{3CC(2)} = (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(J', m')} \sum_{(J'', m'')} (-1)^{J'+J''-J} (-1)^{m'-m''} \\ & \times \hat{C}_{Jm(jj)J''m''(jj)}^{J'm'(jj)} \hat{C}_{J''-m''(jj)\bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\ & \times \left(\frac{2M}{9} \right)^2 \int \frac{d^2 q}{(2\pi)^2} \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{33} \frac{1}{(p+q)^2 + \frac{M^2}{9}(J'' + \frac{1}{3})(J'' + \frac{2}{3})}, \end{aligned} \quad (6.17)$$

$$\begin{aligned} \mathcal{B}_{3,4}^{3CC(2)} = & -\mathcal{B}_{4,3}^{3CC(2)} = (-1)^{m+1} \left(\frac{M}{3} \right)^2 \sum_{(J', m')} \sum_{(J'', m'')} (-1)^{J'+J''-J+1} (-1)^{m'-m''} \\ & \times \hat{C}_{Jm(jj)J''m''(jj)}^{J'm'(jj)} \hat{C}_{J''-m''(jj)\bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\ & \times \frac{4M}{9} \int \frac{d^2 q}{(2\pi)^2} (q_\mu + 2p_\mu) \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu 3} \frac{1}{(p+q)^2 + \frac{M^2}{9}(J'' + \frac{1}{3})(J'' + \frac{2}{3})}. \end{aligned} \quad (6.18)$$

Similarly to the planar case, the off-diagonal parts of $\mathcal{B}_{i,\bar{i}}^{3FF}$ with $(i, \bar{i}) = (3, 4)$ and $(5, 6)$ are exactly zero, and those with $(i, \bar{i}) = (3, 5)$, $(3, 6)$, $(4, 5)$, and $(4, 6)$ turn out to disappear in the Moyal limit, for instance, from Theorem 3 in section 6.2.

6.1.2 $\mathcal{A}_{7,7}$ and $\mathcal{B}_{7,7}$

The planar contributions are

$$\begin{aligned} \mathcal{A}_{7,7}^{4CC} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(J', m')} \sum_{(J'', m'')} (-1)^{m''-m'} \hat{C}_{Jm(jj)\bar{J}\bar{m}(jj)}^{J''m''(jj)} \hat{C}_{J'm'(jj)J'-m'(jj)}^{J''-m''(jj)} \\ & \times \int \frac{d^2 q}{(2\pi)^2} \left\{ \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu\mu} + \frac{4}{q^2 + \frac{M^2}{9}(J' + \frac{1}{3})(J' + \frac{2}{3})} \right\} \end{aligned} \quad (6.19)$$

$$\begin{aligned} \mathcal{A}_{7,7}^{4CD} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(\rho', J', m')} \sum_{(J'', m'')} (-1)^{m''-m'+1} \hat{C}_{Jm(jj)\bar{J}\bar{m}(jj)}^{J''m''(jj)} \hat{\mathcal{D}}_{J'm'(jj)\rho'J'-m'(jj)\rho'}^{J''-m''(jj)} \\ & \times \int \frac{d^2 q}{(2\pi)^2} \tilde{\Delta}_{(\rho', J')}(q)^{-1}, \end{aligned} \quad (6.20)$$

$$\begin{aligned}
\mathcal{A}_{7,7}^{3\mathcal{DD}} = & (-1)^m \left(\frac{M}{3}\right)^2 \sum_{(J',m')} \sum_{(\rho'',J'',m'')} (-1)^{-m'-m''} \\
& \times \left((-1)^{m'} \sqrt{J(J+1)} \hat{\mathcal{D}}_{J''m''(jj)\rho''Jm(jj)0}^{J'-m'(jj)} - (-1)^m \sqrt{J'(J'+1)} \hat{\mathcal{D}}_{J'm'(jj)0J''m''(jj)\rho''}^{J-m(jj)} \right) \\
& \times \left((-1)^{\bar{m}} \sqrt{J'(\bar{J}'+1)} \hat{\mathcal{D}}_{J''-m''(jj)\rho''J'-m'(jj)0}^{\bar{J}-\bar{m}(jj)} - (-1)^{-m'} \sqrt{\bar{J}(\bar{J}+1)} \hat{\mathcal{D}}_{\bar{J}\bar{m}(jj)0J''-m''(jj)\rho''}^{J'-m'(jj)} \right) \\
& \times \frac{M^2}{9} \int \frac{d^2q}{(2\pi)^2} \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{33} \tilde{\Delta}_{(\rho'',J'')}(p+q)^{-1}, \tag{6.21}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{7,7}^{3\mathcal{FF}} = & (-1)^m \left(\frac{M}{3}\right)^2 \sum_{[\kappa',J',m']} \sum_{[\kappa'',J'',m'']} \hat{\mathcal{F}}_{J''m''(jj)\kappa''\bar{J}\bar{m}(jj)}^{J'm'(jj)\kappa'} \hat{\mathcal{F}}_{J'm'(jj)\kappa'Jm(jj)}^{J''m''(jj)\kappa''} \\
& \times \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \text{tr}_8 \left(\hat{\gamma}_7 \hat{D}_{J',\kappa'}(q)^{-1} \hat{\gamma}_7 \hat{D}_{J'',\kappa''}(p+q)^{-1} \right), \tag{6.22}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{7,7}^{3CC(1)} = & (-1)^{m+1} \left(\frac{M}{3}\right)^2 \sum_{(J',m')} \sum_{(J'',m'')} \left[\hat{C}_{J''m''(jj)Jm(jj)}^{J'm'(jj)} \hat{C}_{J'm'(jj)\bar{J}\bar{m}(jj)}^{J''m''(jj)} \right. \\
& \times \int \frac{d^2q}{(2\pi)^2} (q_\mu + 2p_\mu)(q_\nu - p_\nu) \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu 3} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{3\nu} \\
& + (-1)^{m'-m''} \hat{C}_{Jm(jj)J''m''(jj)}^{J'm'(jj)} \hat{C}_{J''-m''(jj)\bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\
& \times \int \frac{d^2q}{(2\pi)^2} (q_\mu + 2p_\mu)(q_\nu + 2p_\nu) \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu\nu} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{33} \left. \right] \\
& + (-1)^{\bar{m}+1} \left(\frac{M}{3}\right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{-m'-m''} \hat{C}_{J''m''(jj)J''m''(jj)}^{J-m(jj)} \hat{C}_{J''-m''(jj)J'-m'(jj)}^{\bar{J}-\bar{m}(jj)} \\
& \times \left(\frac{2M}{9} \right)^2 \left\{ \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \left(\left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{12} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{21} \right. \right. \\
& \quad \left. \left. + \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{21} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{12} \right) \right. \\
& \quad \left. - \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + \frac{M^2}{9}(J' + \frac{1}{3})(J' + \frac{2}{3})} \right. \\
& \quad \left. \times \frac{1}{(p+q)^2 + \frac{M^2}{9}(J'' + \frac{1}{3})(J'' + \frac{2}{3})} \right\}, \tag{6.23}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{7,7}^{3CC(2)} = & \left(\frac{M}{3}\right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{-m'} \hat{C}_{J''m''(jj)J'm'(jj)}^{J-m(jj)} \hat{C}_{\bar{J}\bar{m}(jj)J'-m'(jj)}^{J''m''(jj)} \\
& \times \frac{4M}{9} \int \frac{d^2q}{(2\pi)^2} (q_\mu - p_\mu) \left(\left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{13} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{2\mu} \right. \\
& \quad \left. - \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{23} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{1\mu} \right)
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{(J', m')} \sum_{(J'', m'')} (-1)^{-m' - m''} \hat{C}_{J' m' (jj) J'' m'' (jj)}^{J - m (jj)} \hat{C}_{J'' - m'' (jj) J' - m' (jj)}^{\bar{J} - \bar{m} (jj)} \\
& \times \left(\frac{2M}{9} \right)^2 \int \frac{d^2 q}{(2\pi)^2} \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{11} \left(\tilde{\Delta}_{J''}(-p - q)^{-1} \right)_{22}. \quad (6.24)
\end{aligned}$$

By using (3.39), $\mathcal{A}_{7,7}^{4CD}$ can also be expressed as

$$\begin{aligned}
\mathcal{A}_{7,7}^{4CD} = & (-1)^{\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{(\rho', J', m')} \sum_{(\rho'', J'', m'')} (-1)^{m'' - m'} \\
& \times \hat{\mathcal{D}}_{J'' m'' (jj) \rho'' J' - m' (jj) \rho'}^{J - m (jj)} \hat{\mathcal{D}}_{J' m' (jj) \rho' J'' - m'' (jj) \rho''}^{\bar{J} - \bar{m} (jj)} \int \frac{d^2 q}{(2\pi)^2} \tilde{\Delta}_{(\rho', J')}(q)^{-1}. \quad (6.25)
\end{aligned}$$

The non-planar contributions are

$$\begin{aligned}
\mathcal{B}_{7,7}^{4CC} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(J', m')} \sum_{(J'', m'')} (-1)^{m'' - m' + 1} \hat{C}_{J' m' (jj) J m (jj)}^{J'' m'' (jj)} \hat{C}_{J' - m' (jj) \bar{J} \bar{m} (jj)}^{J'' - m'' (jj)} \\
& \times \int \frac{d^2 q}{(2\pi)^2} \left\{ \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu\mu} + \frac{4}{q^2 + \frac{M^2}{9}(J' + \frac{1}{3})(J' + \frac{2}{3})} \right\}, \quad (6.26)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{7,7}^{4DD} = & (-1)^{\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{(\rho', J', m')} \sum_{(\rho'', J'', m'')} (-1)^{\tilde{Q}' + \tilde{Q}'' - J + 1} (-1)^{m'' - m'} \\
& \times \hat{\mathcal{D}}_{J'' m'' (jj) \rho'' J' - m' (jj) \rho'}^{J - m (jj)} \hat{\mathcal{D}}_{J' - m' (jj) \rho' J'' m'' (jj) \rho''}^{\bar{J} - \bar{m} (jj)} \int \frac{d^2 q}{(2\pi)^2} \tilde{\Delta}_{(\rho', J')}(q)^{-1}. \quad (6.27)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{7,7}^{3DD} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{(J', m')} \sum_{(\rho'', J'', m'')} (-1)^{J' + \tilde{Q}'' - J + 1} (-1)^{-m' - m'' + 1} \\
& \times \left((-1)^{m'} \sqrt{J(J+1)} \hat{\mathcal{D}}_{J'' m'' (jj) \rho'' J m (jj) 0}^{J' - m' (jj)} - (-1)^m \sqrt{J'(J'+1)} \hat{\mathcal{D}}_{J' m' (jj) 0 J'' m'' (jj) \rho''}^{J - m (jj)} \right) \\
& \times \left((-1)^{\bar{m}} \sqrt{J'(\bar{J}+1)} \hat{\mathcal{D}}_{J'' - m'' (jj) \rho'' J' - m' (jj) 0}^{\bar{J} - \bar{m} (jj)} - (-1)^{m'} \sqrt{\bar{J}(\bar{J}+1)} \hat{\mathcal{D}}_{\bar{J} \bar{m} (jj) 0 J'' - m'' (jj) \rho''}^{J' m' (jj)} \right) \\
& \times \frac{M^2}{9} \int \frac{d^2 q}{(2\pi)^2} \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{33} \tilde{\Delta}_{(\rho'', J'')}(p + q)^{-1}, \quad (6.28)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{7,7}^{3FF} = & (-1)^m \left(\frac{M}{3} \right)^2 \sum_{[\kappa', J', m']} \sum_{[\kappa'', J'', m'']} (-1)^{\tilde{U}' + \tilde{U}'' - J + 1} \hat{\mathcal{F}}_{J' m' (jj) \kappa' J m (jj)}^{J'' m'' (jj) \kappa''} \hat{\mathcal{F}}_{J'' m'' (jj) \kappa'' \bar{J} \bar{m} (jj)}^{J' m' (jj) \kappa'} \\
& \times \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \text{tr}_8 \left(\hat{\gamma}_7 \hat{D}_{J', \kappa'}(q)^{-1} \hat{\gamma}_7 \hat{D}_{J'', \kappa''}(p + q)^{-1} \right), \quad (6.29)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{7,7}^{3CC(1)} = & (-1)^{m+1} \left(\frac{M}{3} \right)^2 \sum_{(J', m')} \sum_{(J'', m'')} (-1)^{J' + J'' - J + 1} \left[\hat{C}_{J'' m'' (jj) J m (jj)}^{J' m' (jj)} \hat{C}_{J' m' (jj) \bar{J} \bar{m} (jj)}^{J'' m'' (jj)} \right. \\
& \times \left. \int \frac{d^2 q}{(2\pi)^2} (q_\mu + 2p_\mu)(q_\nu - p_\nu) \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu 3} \left(\tilde{\Delta}_{J''}(-p - q)^{-1} \right)_{3\nu} \right]
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{m'-m''} \hat{C}_{Jm(jj)J''m''(jj)}^{J'm'(jj)} \hat{C}_{J''m''(jj)\bar{J}\bar{m}(jj)}^{J'-m'(jj)} \\
& \times \int \frac{d^2q}{(2\pi)^2} (q_\mu + 2p_\mu)(q_\nu + 2p_\nu) \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{\mu\nu} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{33} \Bigg] \\
& + (-1)^{\bar{m}+1} \left(\frac{M}{3} \right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{-m'-m''} \hat{C}_{J'm'(jj)J''m''(jj)}^{J-m(jj)} \hat{C}_{J'-m'(jj)J''-m''(jj)}^{\bar{J}-\bar{m}(jj)} \\
& \times \left(\frac{2M}{9} \right)^2 \left\{ \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \left(\left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{11} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{22} \right. \right. \\
& \quad \left. \left. + \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{22} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{11} \right) \right. \\
& \quad \left. + \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + \frac{M^2}{9}(J' + \frac{1}{3})(J' + \frac{2}{3})} \right. \\
& \quad \left. \times \frac{1}{(p+q)^2 + \frac{M^2}{9}(J'' + \frac{1}{3})(J'' + \frac{2}{3})} \right\}, \quad (6.30)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{7,7}^{3CC(2)} &= \left(\frac{M}{3} \right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{J'+J''-J+1} (-1)^{-m'} \hat{C}_{J''m''(jj)J'm'(jj)}^{J-m(jj)} \hat{C}_{\bar{J}\bar{m}(jj)J'-m'(jj)}^{J''m''(jj)} \\
& \times \frac{4M}{9} \int \frac{d^2q}{(2\pi)^2} (q_\mu - p_\mu) \left(\left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{13} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{2\mu} \right. \\
& \quad \left. - \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{23} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{1\mu} \right) \\
& + (-1)^{\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{(J',m')} \sum_{(J'',m'')} (-1)^{J'+J''-J} (-1)^{-m'-m''} \\
& \quad \times \hat{C}_{J'm'(jj)J''m''(jj)}^{J-m(jj)} \hat{C}_{J''-m''(jj)J'-m'(jj)}^{\bar{J}-\bar{m}(jj)} \\
& \times \left(\frac{2M}{9} \right)^2 \int \frac{d^2q}{(2\pi)^2} \left(\tilde{\Delta}_{J'}(q)^{-1} \right)_{12} \left(\tilde{\Delta}_{J''}(-p-q)^{-1} \right)_{21}. \quad (6.31)
\end{aligned}$$

6.2 Strategy to evaluate the one-loop diagrams

In the following, we compute the one-loop diagrams listed in (6.4)–(6.31) in the Moyal limit of Step 1:

$$M \rightarrow 0, \quad n = 2j + 1 \rightarrow \infty, \quad \frac{1}{n(M/3)^2} \equiv \frac{\Theta}{2} : \text{fixed}. \quad (6.32)$$

As in the computation of the one-point functions, UV divergences from the momentum integrations are regularized by the UV cutoff Λ_p ⁴. Even after setting the cutoff, we often

⁴ As we will see, Λ_p -dependent terms eventually cancel each other and thus the gauge symmetry is not spoiled due to the cutoff. Alternatively, we can see at least at the one-loop level that the cutoff

encounter divergences in the sums over J' and J'' from the region of $J' = 0$ or $J'' = 0$. Since they are angular momenta in the fuzzy S^2 , they can be regarded as “IR” divergences on the S^2 . We consider the case with the external angular momentum J nonzero such that the Moyal limit of the momentum $u \equiv \frac{M}{3} \cdot J$ is kept finite as $M \rightarrow 0$. The external momentum $|p|$ in the original \mathbb{R}^2 is assumed to be of the same order as u . From the triangular inequality

$$|J' - J''| \leq J \leq J' + J'', \quad (6.33)$$

the IR divergences may arise when one of J' and J'' (say J'') is equal to zero. We first remove the region of $J'' = 0$ from the summation in that case, and call the remaining part the “UV part”. The $J'' = 0$ part is treated separately by introducing the IR cutoff δ , which is removed after taking the Moyal limit (6.32). By using the expression of the vertex coefficients (B.23), (B.95), and (B.112), we can carry out the sum over variables other than J' and J'' in the UV part of the planar contributions (6.4)–(6.11) and (6.19)–(6.25). The result of each contribution is expressed as a sum of the following building block:

$$\begin{aligned} \mathcal{A}^{\text{UV}} \equiv & \left(\frac{M}{3}\right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n f(J', J''; J) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ & \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{g(\tilde{p}, \tilde{q})}{M_{a,b} \left(\tilde{P}_i(J'); \tilde{Q}_k(J''); \tilde{q}, \tilde{p} \right)}. \end{aligned} \quad (6.34)$$

Similarly, for the non-planar contribution in (6.12)–(6.18) and (6.26)–(6.31), the corresponding building block takes the form ⁵

$$\begin{aligned} \mathcal{B}^{\text{UV}} \equiv & - \left(\frac{M}{3}\right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n (-1)^{J'+J''+J} f(J', J''; J) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ & \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{g(\tilde{p}, \tilde{q})}{M_{a,b} \left(\tilde{P}_i(J'); \tilde{Q}_k(J''); \tilde{q}, \tilde{p} \right)}. \end{aligned} \quad (6.35)$$

We have rescaled the external and internal momenta p_μ and q_μ as $p_\mu = \left(\frac{M}{3}\right) \tilde{p}_\mu$ and $q_\mu = \left(\frac{M}{3}\right) \tilde{q}_\mu$. Correspondingly, we have rescaled the UV cutoff as $\Lambda_p = \left(\frac{M}{3}\right) \tilde{\Lambda}_p$ and the momentum integral $\int^{\tilde{\Lambda}_p}$ denotes the integration with the rescaled UV cutoff. $f(J', J''; J)$

regularization corresponds to the dimensional reduction regularization ($d = 2 - 2\epsilon$) which respects the gauge symmetry, by $\ln \Lambda_p^2 \Leftrightarrow \frac{1}{\epsilon} - \gamma + \ln(4\pi)$ with γ being the Euler constant.

⁵Note that $\mathcal{A}_{i,i}^{4C\mathcal{D}}$ corresponds to $\mathcal{B}_{i,i}^{4D\mathcal{D}}$ from the expressions (6.11) and (6.13). This is also similar for $\mathcal{A}_{7,7}^{4C\mathcal{D}}$ and $\mathcal{B}_{7,7}^{4D\mathcal{D}}$.

is a function of the form

$$f(J', J''; J) = C J^{N_J} (J' - J'')^{N_\Delta} (J')^{N_1} (J'')^{N_2}, \quad (6.36)$$

where C is an $\mathcal{O}(1)$ constant, and N_J , N_Δ , N_1 and N_2 are integers. $g(\tilde{p}, \tilde{q})$ is a homogeneous polynomial of \tilde{p} and \tilde{q} consisting of \tilde{q}^2 , \tilde{p}^2 , $\tilde{p} \cdot \tilde{q}$, $(\tilde{p} \times \tilde{q})^2$. $P_i(J) = (\frac{M}{3})^2 \tilde{P}_i(J)$ ($i = 1, \dots, a$) and $Q_k(J) = (\frac{M}{3})^2 \tilde{Q}_k(J)$ ($k = 1, \dots, b$) are monic quadratic polynomials of J with $0 \leq J \leq 2j$. The symbol $M_{a,b}(\tilde{P}_i(J'); \tilde{Q}_k(J''); \tilde{q}, \tilde{p})$ is defined as

$$\begin{aligned} M_{a,b}(\tilde{P}_i(J'); \tilde{Q}_k(J''); \tilde{q}, \tilde{p}) &\equiv \prod_{i=1}^a \left(\tilde{q}^2 + \tilde{P}_i(J') \right) \cdot \prod_{k=1}^b \left((\tilde{q} + \tilde{p})^2 + \tilde{Q}_k(J'') \right) \\ &= \left(\frac{M}{3} \right)^{-2(a+b)} \prod_{i=1}^a (q^2 + P_i(J')) \cdot \prod_{k=1}^b ((q + p)^2 + Q_k(J'')). \end{aligned} \quad (6.37)$$

For the case of $b = 0$, we have

$$M_{a,0}(\tilde{P}_i(J'); \tilde{q}) = \prod_{i=1}^a \left(\tilde{q}^2 + \tilde{P}_i(J') \right) = \left(\frac{M}{3} \right)^{-2a} \prod_{i=1}^a (q^2 + P_i(J')). \quad (6.38)$$

In the calculation, the function

$$\begin{aligned} \hat{L}(\tilde{P}(J'), \tilde{Q}(J''); \tilde{p}) &\equiv \int \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{4\pi}{M_{1,1}(\tilde{P}(J'), \tilde{Q}(J''); \tilde{q}, \tilde{p})} \\ &= \frac{1}{\sqrt{(\tilde{p}^2)^2 + 2(\tilde{P}(J') + \tilde{Q}(J''))\tilde{p}^2 + (\tilde{P}(J') - \tilde{Q}(J''))^2}} \\ &\quad \times \ln \left(\frac{\tilde{p}^2 + \tilde{P}(J') + \tilde{Q}(J'') + \sqrt{(\tilde{p}^2)^2 + 2(\tilde{P}(J') + \tilde{Q}(J''))\tilde{p}^2 + (\tilde{P}(J') - \tilde{Q}(J''))^2}}{\tilde{p}^2 + \tilde{P}(J') + \tilde{Q}(J'') - \sqrt{(\tilde{p}^2)^2 + 2(\tilde{P}(J') + \tilde{Q}(J''))\tilde{p}^2 + (\tilde{P}(J') - \tilde{Q}(J''))^2}} \right) \end{aligned} \quad (6.39)$$

is used, and the polynomials

$$\begin{aligned} \tilde{A}(J) &\equiv (J + \frac{1}{3})(J + \frac{2}{3}), \quad \tilde{B}(J) \equiv J(J + \frac{1}{3}), \quad \tilde{C}(J) \equiv (J + 1)(J + \frac{2}{3}), \\ \tilde{D}(J) &\equiv J(J + 1), \quad \tilde{E}(J) \equiv (J + 1)^2, \quad \tilde{F}(J) \equiv J^2 \end{aligned} \quad (6.40)$$

appear as $\tilde{P}_i(J)$ and $\tilde{Q}_k(J)$. For $\mathcal{A}_{7,7}^{3CC(1)}$, $\mathcal{A}_{7,7}^{3CC(2)}$ and $\mathcal{B}_{7,7}^{3CC(1)}$, $\mathcal{B}_{7,7}^{3CC(2)}$, the expressions (6.34) and (6.35) hold respectively up to additive terms that are irrelevant in the limit (6.32). It should be noted that the summand of (6.35) is different from that of (6.34) just by the sign factor $-(-1)^{J'+J''+J}$.

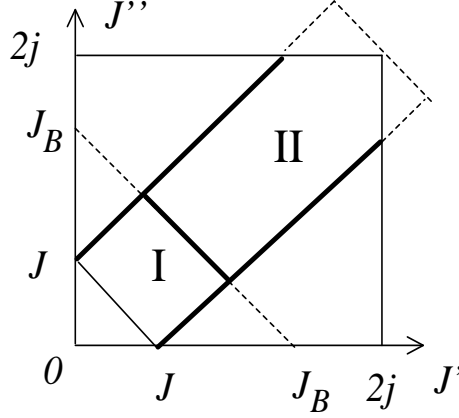


Figure 1: Regions I and II are surrounded by solid lines. Note that the two triangular regions at the top right give no contribution since the $6j$ symbol is null there.

Note that the $6j$ symbol $\left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}$ vanishes outside the region of $J \leq J' + J'' \leq 4j$ and $|J' - J''| \leq J$. In order to estimate the expressions (6.34) and (6.35) in the limit (6.32), we separate the range of the summation of angular momenta J' and J'' ,

$$0 \leq J' \leq 2j, \quad 1 \leq J'' \leq 2j, \quad -J \leq J' - J'' \leq J, \quad (6.41)$$

into two regions satisfying

$$\begin{cases} \text{Region I :} & J \leq J' + J'' < J_B \\ \text{Region II :} & J_B \leq J' + J'' \leq 4j \end{cases} \quad \left(J_B = \mathcal{O}(M^{-2\alpha}), \quad \frac{1}{2} < \alpha < 1 \right). \quad (6.42)$$

The regions are depicted in Fig. 1. In Region I, we can evaluate the summations of J' and J'' by integrations in the Moyal limit, but this is not justified in Region II. Since J is typically of the order of $\mathcal{O}(M^{-1})$, $J \ll J_B \ll j$ in the limit $M \rightarrow 0$. Correspondingly, we divide \mathcal{A}^{UV} into two parts:

$$\begin{aligned} \mathcal{A}_I^{\text{UV}} \equiv & \left(\frac{M}{3} \right)^2 \sum_{J', J'' \in \text{Region I}} n f(J', J'', J) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ & \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{g(\tilde{p}, \tilde{q})}{M_{a,b} \left(\tilde{P}_i(J'); \tilde{Q}_k(J''); \tilde{q}, \tilde{p} \right)}, \end{aligned} \quad (6.43)$$

$$\mathcal{A}_{II}^{\text{UV}} \equiv \left(\frac{M}{3}\right)^2 \sum_{J', J'' \in \text{Region II}} n f(J', J'', J) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{g(\tilde{p}, \tilde{q})}{M_{a,b}(\tilde{P}_i(J'); \tilde{Q}_k(J''); \tilde{q}, \tilde{p})}; \quad (6.44)$$

we do the same for \mathcal{B}^{UV} .

Here the amount of computation is considerably reduced by looking at the asymptotic behavior of the integrands. We assume that the homogeneous polynomial $g(\tilde{p}, \tilde{q})$ behaves asymptotically as

$$g(\tilde{p}, \tilde{q}) \sim |\tilde{p}|^{n_p} |\tilde{q}|^{n_q} (1 + \mathcal{O}(|\tilde{p}||\tilde{q}|^{-1})) \quad (6.45)$$

for $|\tilde{q}| \gg 1$. Then we can claim the following theorem with respect to the summation in Region II:

Theorem 1 (Region II)

Let us define

$$w \equiv N_1 + N_2 - 2(a + b) + n_q + 2. \quad (6.46)$$

Then $\mathcal{A}_{II}^{\text{UV}}$ vanishes in the Moyal limit (6.32) if one of the following conditions is satisfied;

1. $w > 0$ and $W^+ \equiv -N_J - N_\Delta - n_p + 2 - 2w > 0$
2. $w \leq 0$ and $W^- \equiv -N_J - N_\Delta - n_p + 2 - w > 0$.

In Region I, on the other hand, the summation over J' and J'' can be approximated by an integration over the variables $u' \equiv \frac{M}{3} \cdot J'$ and $u'' \equiv \frac{M}{3} \cdot J''$ when M is sufficiently small. Under the assumption that $\mathcal{A}_{II}^{\text{UV}}$ vanishes in the Moyal limit, the IR property of the function ⁶ $g(p, q - tp)$ has a key role, as discussed in Appendix D.2. In the expansion

$$g(p, q - tp) = \sum_{\ell \geq 0} \alpha_\ell(t) q^{2\ell}, \quad (6.47)$$

the coefficient $\alpha_\ell(t)$, which is a polynomial of t , is assumed to behave as

$$\alpha_\ell(t) = \begin{cases} t^{a_\ell^{(0)}} + \mathcal{O}(t^{a_\ell^{(0)}+1}) & (t \sim 0) \\ (1-t)^{a_\ell^{(1)}} + \mathcal{O}((1-t)^{a_\ell^{(1)}+1}) & (t \sim 1) \end{cases} \quad (6.48)$$

up to multiplicative $\mathcal{O}(1)$ factors. Then we can prove the theorem with respect to the summation over Region I:

⁶ The argument $q - tp$ arises by introducing the Feynman parameter t .

Theorem 2 (in Region I)

Suppose that $\mathcal{A}_{II}^{\text{UV}}$ vanishes in the Moyal limit by satisfying the condition of Theorem

1. In the case of $a, b \geq 1$, $\mathcal{A}_I^{\text{UV}}$ vanishes in the same limit if both of

$$D_0 \equiv W^- + N_1 - 2a + 3 + 2 \min_{\ell}(\ell + a_{\ell}^{(0)}) > 0 \quad (6.49)$$

and

$$D_1 \equiv W^- + N_2 - 2b + 3 + 2 \min_{\ell}(\ell + a_{\ell}^{(1)}) > 0 \quad (6.50)$$

are satisfied. When one of a and b is zero (say $b = 0$), the condition for $\mathcal{A}_I^{\text{UV}} \rightarrow 0$ is given by

$$D_0^{(b=0)} \equiv W^- + N_1 - 2a + 2\ell_0 + 3 > 0, \quad (6.51)$$

where ℓ_0 is the smallest ℓ such that $\alpha_{\ell}(0) \neq 0$.

Proofs of these theorems are given in Appendix D.

As we mentioned above, when the planar contribution has the form (6.34), the corresponding non-planar contribution takes the form (6.35). In Region I, where $J, J', J'' \ll j$ is satisfied, the Wigner $6j$ symbol can be approximated as (D.21). Since it is negligible for $J' + J'' + J$ odd, the phase factor $(-1)^{J'+J''+J}$ in (6.35) is irrelevant in Region I. In addition, looking at the proof in Appendix D.1, we can evaluate the non-planar contribution in exactly the same way as the planar contribution because the difference between them is only the factor $(-1)^{J'+J''+J}$. Namely, $|\mathcal{B}_{II}^{\text{UV}}|$ can also be bounded from the above by the r.h.s. of (D.17). Therefore, we can claim

Theorem 3

When $\mathcal{A}_I^{\text{UV}}$ and $\mathcal{A}_{II}^{\text{UV}}$ vanish in the Moyal limit, the corresponding $\mathcal{B}_I^{\text{UV}}$ and $\mathcal{B}_{II}^{\text{UV}}$ also vanish in the same limit.

These three theorems are useful in evaluating (6.4)–(6.31).

6.3 Contributions from one-loop diagrams

We explicitly calculate the diagrams listed in section 6.1. Although there are a number of diagrams, applying the theorems avoids carrying out brute-force computation for all of them. We first look at the diagrams that concern to the two-point function $\langle x_i x_i \rangle$ ($i = 3, \dots, 6$), where the computations of (6.4), (6.12), (6.8), and (6.16) are presented as typical examples. For the other diagrams, we simply show the results. After that, it is shown that the diagrams for $\langle x_7 x_7 \rangle$ give identical results with those for $\langle x_i x_i \rangle$ ($i = 3, \dots, 6$) in the Moyal limit.

6.3.1 Diagrams from 4-point vertices: $\mathcal{A}_{i,i}^{4CC}$ and $\mathcal{B}_{i,i}^{4CC}$

Let us first look at the planar contribution (6.4). By using

$$\sum_{m'=-J'}^{J'} (-1)^{-m'} C_{J' m' J' -m'}^{J' -m''} = (-1)^{-J} \sqrt{2J'+1} \delta_{J'' 0} \delta_{m'' 0}, \quad (6.52)$$

$$\begin{Bmatrix} J_1 & J_2 & 0 \\ j & j & j \end{Bmatrix} = (-1)^{J_1+2j} \frac{\delta_{J_1 J_2}}{\sqrt{n(2J_1+1)}}, \quad (6.53)$$

the UV part can be written as

$$\begin{aligned} \mathcal{A}_{i,i}^{4CC,UV} = & \delta_{J \bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n(2J'+1)(2J''+1) \begin{Bmatrix} J' & J'' & J \\ j & j & j \end{Bmatrix}^2 \\ & \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \left\{ \frac{2}{\tilde{q}^2 + \tilde{D}(J'')} + \frac{4}{\tilde{q}^2 + \tilde{A}(J'')} \right. \\ & \left. - \frac{4}{9} \frac{\tilde{q}^2}{M_{3,0}(\tilde{B}(J''), \tilde{C}(J''), \tilde{D}(J''); \tilde{q})} - \frac{4}{9} \frac{\tilde{q}^2}{M_{3,0}(\tilde{A}(J''), \tilde{B}(J''), \tilde{C}(J''); \tilde{q})} \right\}. \end{aligned} \quad (6.54)$$

For example, $M_{3,0}(\tilde{B}(J''), \tilde{C}(J''), \tilde{D}(J''); \tilde{q})$ stands for (6.38) with $a = 3$, $\tilde{P}_1 = \tilde{B}$, $\tilde{P}_2 = \tilde{C}$, and $\tilde{P}_3 = \tilde{D}$. As a result of straightforward calculations together with the identity

$$\sum_{J'=0}^{2j} n(2J'+1) \begin{Bmatrix} J' & J'' & J \\ j & j & j \end{Bmatrix}^2 = 1, \quad (6.55)$$

it turns out that the last two terms do not contribute in the Moyal limit, and we have

$$\mathcal{A}_{i,i}^{4CC,UV} = \delta_{J \bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \sum_{J''=1}^{2j} (2J''+1) \left(6 \ln \tilde{\Lambda}_p^2 - 2 \ln \tilde{D}(J'') - 4 \ln \tilde{A}(J'') \right) + \dots \quad (6.56)$$

Here and in what follows, the ellipsis (\dots) expresses terms that vanish in the Moyal limit ⁷.

For the IR part of (6.4) (contribution from $J'' = 0$), introducing the IR cutoff $J'' = \delta$ ($0 < \delta \ll 1$) regularizes it as

$$\mathcal{A}_{i,i}^{4CC,IR} = \delta_{J \bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} n(2J'+1) \begin{Bmatrix} J & J' & \delta \\ j & j & j \end{Bmatrix}^2$$

⁷ The last two terms ($a = 3, b = 0$) have $(N_1, N_2) = (1, 1), (1, 0), (0, 1), (0, 0)$, $N_J = N_\Delta = 0$, $(n_p, n_q) = (0, 2)$, $\ell_0 = 1$, which leads to $D_0^{(b=0)} \geq 2 > 0$. Thus, Theorems 1 and 2 also tell that the terms are negligible in the Moyal limit.

$$\begin{aligned}
& \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \left\{ \frac{2}{\tilde{q}^2 + \tilde{D}(\delta)} + \frac{4}{\tilde{q}^2 + \tilde{A}(\delta)} \right. \\
& \quad \left. - \frac{4}{9} \frac{\tilde{q}^2}{M_{3,0}(\tilde{B}(\delta), \tilde{C}(\delta), \tilde{D}(\delta); \tilde{q})} - \frac{4}{9} \frac{\tilde{q}^2}{M_{3,0}(\tilde{A}(\delta), \tilde{B}(\delta), \tilde{C}(\delta); \tilde{q})} \right\} \\
& = \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \left\{ 6h_j^{(p)}(\delta) \ln \tilde{\Lambda}_p^2 - \frac{4}{3} \ln \delta \right\} + \dots
\end{aligned} \tag{6.57}$$

with

$$h_j^{(p)}(\delta) \equiv (1 + 2\delta) \sum_{J'=0}^{2j} n(2J' + 1) \left\{ \begin{matrix} J & J' & \delta \\ j & j & j \end{matrix} \right\}^2. \tag{6.58}$$

We have assumed

$$\left\{ \begin{matrix} J & J' & \delta \\ j & j & j \end{matrix} \right\} = \left\{ \begin{matrix} J & J' & 0 \\ j & j & j \end{matrix} \right\} + o(\delta^0), \tag{6.59}$$

and used (6.53).

We next evaluate the non-planar contribution (6.12). Theorem 3 leads to its UV part as

$$\begin{aligned}
\mathcal{B}_{i,i}^{4CC,UV} &= -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} (-1)^{J'+J''+J} n(2J' + 1)(2J'' + 1) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\
& \times \left(6 \ln \tilde{\Lambda}_p^2 - 2 \ln \tilde{D}(J'') - 4 \ln \tilde{A}(J'') \right) + \dots
\end{aligned} \tag{6.60}$$

Similarly to (6.57), the IR part of (6.12) becomes

$$\begin{aligned}
\mathcal{B}_{i,i}^{4CC,IR} &= -\delta_{J\bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} (-1)^{J+J'+\delta} n(2J' + 1)(1 + 2\delta) \left\{ \begin{matrix} J & J' & \delta \\ j & j & j \end{matrix} \right\}^2 \\
& \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \left\{ \frac{2}{\tilde{q}^2 + \tilde{D}(\delta)} + \frac{4}{\tilde{q}^2 + \tilde{A}(\delta)} \right. \\
& \quad \left. - \frac{4}{9} \frac{\tilde{q}^2}{M_{3,0}(\tilde{B}(\delta), \tilde{C}(\delta), \tilde{D}(\delta); \tilde{q}, \tilde{p})} - \frac{4}{9} \frac{\tilde{q}^2}{M_{3,0}(\tilde{A}(\delta), \tilde{B}(\delta), \tilde{C}(\delta); \tilde{q}, \tilde{p})} \right\} \\
& = -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \left\{ 6h_j^{(np)}(\delta) \ln \tilde{\Lambda}_p^2 - \frac{4}{3} \ln \delta \right\} + \dots
\end{aligned} \tag{6.61}$$

with

$$h_j^{(np)}(\delta) \equiv (1 + 2\delta) \sum_{J'=0}^{2j} (-1)^{J+J'+\delta} n(2J' + 1) \left\{ \begin{matrix} J & J' & \delta \\ j & j & j \end{matrix} \right\}^2. \tag{6.62}$$

6.3.2 Diagrams from 3-point vertices: $\mathcal{A}_{i,i}^{3CC(1)}$ and $\mathcal{B}_{i,i}^{3CC(1)}$

The UV part of the planar contribution (6.8) can be expressed as

$$\begin{aligned} \mathcal{A}_{i,i}^{3CC(1),\text{UV}} &= \delta_{J\bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n(2J'+1)(2J''+1) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ &\times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \left\{ -\frac{(\tilde{q}-\tilde{p})^2}{M_{1,1}(\tilde{A}(J'); \tilde{D}(J''); \tilde{q}, \tilde{p})} + \frac{16}{9} \frac{(\tilde{q} \times \tilde{p})^2}{M_{1,3}(\tilde{A}(J'); \tilde{B}(J''), \tilde{C}(J''), \tilde{D}(J''); \tilde{q}, \tilde{p})} \right\}. \end{aligned} \quad (6.63)$$

It is easy to see that the second term in the integrand does not contribute in the Moyal limit due to Theorems 1 and 2. For the first term, we rewrite the numerator as

$$(\tilde{q}-\tilde{p})^2 = -\left((\tilde{q}+\tilde{p})^2 + \tilde{D}(J'')\right) + 2\left(\tilde{q}^2 + \tilde{A}(J')\right) + 2\tilde{p}^2 + \tilde{D}(J'') - 2\tilde{A}(J'), \quad (6.64)$$

and apply the theorems, leading to

$$\begin{aligned} \mathcal{A}_{i,i}^{3CC(1),\text{UV}} &= \delta_{J\bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ &\times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \left[(2J'+1)(2J''+1) \left\{ \frac{1}{\tilde{q}^2 + \tilde{A}(J')} - \frac{2}{\tilde{q}^2 + \tilde{D}(J'')} \right\} \right. \\ &\quad \left. + \frac{4J'J''(-2\tilde{p}^2 + 2J'^2 - J''^2)}{M_{1,1}(\tilde{A}(J'), \tilde{D}(J''); \tilde{q}, \tilde{p})} \right] + \dots \\ &= \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \left[\sum_{J''=1}^{2j} (2J''+1) \left(-\ln \tilde{\Lambda}_p^2 - \ln \tilde{A}(J'') + 2 \ln \tilde{D}(J'') \right) \right. \\ &\quad \left. + \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 4J'J''(-2\tilde{p}^2 + 2J'^2 - J''^2) \hat{L}(\tilde{A}(J'), \tilde{D}(J''); \tilde{p}) \right] \\ &+ \dots, \end{aligned} \quad (6.65)$$

where the function $\hat{L}(\tilde{A}, \tilde{D}; \tilde{p})$ is given by (6.39).

The IR part of (6.8) from $J'' = 0$ is regularized by the IR cutoff δ and becomes

$$\begin{aligned} \mathcal{A}_{i,i}^{3CC(1),\text{IR}} &= \delta_{J\bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} n(2J'+1)(1+2\delta) \left\{ \begin{matrix} J & J' & \delta \\ j & j & j \end{matrix} \right\}^2 \\ &\times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \left\{ -\frac{(\tilde{q}-\tilde{p})^2}{M_{1,1}(\tilde{A}(J'); \tilde{D}(\delta); \tilde{q}, \tilde{p})} + \frac{16}{9} \frac{(\tilde{q} \times \tilde{p})^2}{M_{1,3}(\tilde{A}(J'); \tilde{B}(\delta), \tilde{C}(\delta), \tilde{D}(\delta); \tilde{q}, \tilde{p})} \right\} \end{aligned}$$

$$= \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \left\{ -h_j^{(p)}(\delta) \ln \tilde{\Lambda}_p^2 + \frac{4}{3} \left(\frac{\tilde{p}^2 - \tilde{A}(J)}{\tilde{p}^2 + \tilde{A}(J)} + 1 \right) \ln \delta \right\} + \dots \quad (6.66)$$

For the non-planar contribution (6.16), Theorem 3 gives its UV part as

$$\begin{aligned} \mathcal{B}_{i,i}^{3CC(1),UV} &= -\delta_{J\bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} (-1)^{J'+J''+J} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ &\quad \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \left[(2J'+1)(2J''+1) \left\{ \frac{1}{\tilde{q}^2 + \tilde{A}(J')} - \frac{2}{\tilde{q}^2 + \tilde{D}(J'')} \right\} \right. \\ &\quad \left. + \frac{4J'J''(-2\tilde{p}^2 + 2J'^2 - J''^2)}{M_{1,1}(\tilde{A}(J'), \tilde{D}(J''); \tilde{q}, \tilde{p})} \right] + \dots \\ &= -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} (-1)^{J'+J''+J} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ &\quad \times \left[(2J'+1)(2J''+1) \left\{ -\ln \tilde{\Lambda}_p^2 - \ln \tilde{A}(J') + 2 \ln \tilde{D}(J'') \right\} \right. \\ &\quad \left. + 4J'J''(-2\tilde{p}^2 + 2J'^2 - J''^2) \hat{L}(\tilde{A}(J'), \tilde{D}(J''); \tilde{p}) \right] + \dots \quad (6.67) \end{aligned}$$

We have the IR part of (6.16):

$$\begin{aligned} \mathcal{B}_{i,i}^{3CC(1),IR} &= -\delta_{J\bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} (-1)^{J+J'+\delta} n(2J'+1)(1+2\delta) \left\{ \begin{matrix} J & J' & \delta \\ j & j & j \end{matrix} \right\}^2 \\ &\quad \times \int^{\tilde{\Lambda}_p} \frac{d^2 \tilde{q}}{(2\pi)^2} \left\{ -\frac{(\tilde{q} - \tilde{p})^2}{M_{1,1}(\tilde{A}(J'); \tilde{D}(\delta); \tilde{q}, \tilde{p})} + \frac{16}{9} \frac{(\tilde{q} \times \tilde{p})^2}{M_{1,3}(\tilde{A}(J'); \tilde{B}(\delta), \tilde{C}(\delta), \tilde{D}(\delta); \tilde{q}, \tilde{p})} \right\} \\ &= -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \left\{ -h_j^{(np)}(\delta) \ln \tilde{\Lambda}_p^2 + \frac{4}{3} \left(\frac{\tilde{p}^2 - \tilde{A}(J)}{\tilde{p}^2 + \tilde{A}(J)} + 1 \right) \ln \delta \right\} + \dots \quad (6.68) \end{aligned}$$

6.3.3 Other diagrams for $\langle x_i x_i \rangle$ ($i = 3, \dots, 6$)

Here we present the results of other one-loop diagrams for $\langle x_i x_i \rangle$ ($i = 3, \dots, 6$), except for $\mathcal{A}^{3CC(2)}$ and $\mathcal{B}^{2CC(2)}$. The results of the UV parts of the planar diagrams are expressed as

$$\begin{aligned} \mathcal{A}_{i,i}^{4CD,UV} &= \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \sum_{J''=1}^{2j} \left\{ 3(2J''+1) \ln \tilde{\Lambda}_p^2 - (2J''+1) \ln \tilde{D}(J'') \right. \\ &\quad \left. - (2J''+3) \ln \tilde{E}(J'') - (2J''-1) \ln \tilde{F}(J'') \right\} \end{aligned}$$

$$+ \dots, \quad (6.69)$$

$$\begin{aligned} \mathcal{A}_{i,i}^{3\mathcal{DD},\text{UV}} = & \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \left(-\frac{2J'}{J''}\right) \\ & \times \left[2(J'^2 - J^2)^2 \widehat{L}(\tilde{A}(J'), \tilde{D}(J''); \tilde{p}) \right. \\ & \left. + ((J' + J'')^2 - J^2) (J^2 - (J' - J'')^2) \left\{ \widehat{L}(\tilde{A}(J'), \tilde{E}(J''); \tilde{p}) + \widehat{L}(\tilde{A}(J'), \tilde{F}(J''); \tilde{p}) \right\} \right] \\ & + \dots, \quad (6.70) \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{i,i}^{3\mathcal{FF},\text{UV}} = & \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \left[\sum_{J''=1}^{2j} \left\{ -8(2J'' + 1) \ln \tilde{\Lambda}_p^2 + 4(2J'' + 1) \ln \tilde{A}(J'') \right. \right. \\ & \left. \left. + 2(J'' + 1) \left(\ln \tilde{C}(J'') + \ln \tilde{D}(J'') \right) + 2J'' \left(\ln \tilde{B}(J'') + \ln \tilde{F}(J'') \right) \right\} \right. \\ & + \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 4J'J'' (\tilde{p}^2 + J^2) \left\{ \widehat{L}(\tilde{A}(J'), \tilde{B}(J''); \tilde{p}) \right. \\ & \left. \left. + \widehat{L}(\tilde{A}(J'), \tilde{C}(J''); \tilde{p}) + \widehat{L}(\tilde{A}(J'), \tilde{E}(J''); \tilde{p}) + \widehat{L}(\tilde{A}(J'), \tilde{F}(J''); \tilde{p}) \right\} \right] \\ & + \dots. \quad (6.71) \end{aligned}$$

The IR parts of the planar contributions are

$$\mathcal{A}_{i,i}^{4\mathcal{CD},\text{IR}} = \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \left(3h_j^{(\text{p})}(\delta) \ln \tilde{\Lambda}_p^2 \right) + \dots, \quad (6.72)$$

$$\mathcal{A}_{i,i}^{3\mathcal{DD},\text{IR}} = \dots, \quad (6.73)$$

$$\mathcal{A}_{i,i}^{3\mathcal{FF},\text{IR}} = \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \left(-8h_j^{(\text{p})}(\delta) \ln \tilde{\Lambda}_p^2 \right) + \dots. \quad (6.74)$$

For the non-planar diagrams, the results of the UV parts are obtained as

$$\begin{aligned} \mathcal{B}_{i,i}^{4\mathcal{CD},\text{UV}} = & -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} (-1)^{J'+J''+J} n(2J' + 1) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ & \times \left\{ 3(2J'' + 1) \ln \tilde{\Lambda}_p^2 - (2J'' + 1) \ln \tilde{D}(J'') \right. \\ & \left. - (2J'' + 3) \ln \tilde{E}(J'') - (2J'' - 1) \ln \tilde{F}(J'') \right\} \\ & + \dots, \quad (6.75) \end{aligned}$$

$$\begin{aligned} \mathcal{B}_{i,i}^{3\mathcal{DD},\text{UV}} = & -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} (-1)^{J'+J''+J} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \left(-\frac{2J'}{J''}\right) \\ & \times \left[2(J'^2 - J^2)^2 \widehat{L}(\tilde{A}(J'), \tilde{D}(J''); \tilde{p}) \right. \end{aligned}$$

$$+ ((J' + J'')^2 - J^2) (J^2 - (J' - J'')^2) \left\{ \widehat{L}(\widetilde{A}(J'), \widetilde{E}(J''); \widetilde{p}) + \widehat{L}(\widetilde{A}(J'), \widetilde{F}(J''); \widetilde{p}) \right\} \\ + \dots, \quad (6.76)$$

$$\mathcal{B}_{i,i}^{3\mathcal{FF},\text{UV}} = -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} (-1)^{J'+J''+J} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ \times \left[-8(2J' + 1)(2J'' + 1) \ln \widetilde{\Lambda}_p^2 + 4(2J' + 1)(2J'' + 1) \ln \widetilde{A}(J'') \right. \\ + 2(2J' + 1)(J'' + 1) \left(\ln \widetilde{C}(J'') + \ln \widetilde{E}(J'') \right) \\ + 2(2J' + 1)J'' \left(\ln \widetilde{B}(J'') + \ln \widetilde{F}(J'') \right) \\ + 4J'J'' (\widetilde{p}^2 + J^2) \left\{ \widehat{L}(\widetilde{A}(J'), \widetilde{B}(J''); \widetilde{p}) + \widehat{L}(\widetilde{A}(J'), \widetilde{C}(J''); \widetilde{p}) \right. \\ \left. \left. + \widehat{L}(\widetilde{A}(J'), \widetilde{E}(J''); \widetilde{p}) + \widehat{L}(\widetilde{A}(J'), \widetilde{F}(J''); \widetilde{p}) \right\} \right] \\ + \dots. \quad (6.77)$$

The IR parts are given by

$$\mathcal{B}_{i,i}^{4\mathcal{CD},\text{IR}} = -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \left(3h_j^{(\text{np})}(\delta) \ln \widetilde{\Lambda}_p^2 \right) + \dots, \quad (6.78)$$

$$\mathcal{B}_{i,i}^{3\mathcal{DD},\text{IR}} = \dots, \quad (6.79)$$

$$\mathcal{B}_{i,i}^{3\mathcal{FF},\text{IR}} = -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3} \right)^2 \left(-8h_j^{(\text{np})}(\delta) \ln \widetilde{\Lambda}_p^2 \right) + \dots. \quad (6.80)$$

6.3.4 Diagrams for $\langle x_7 x_7 \rangle$ and extra diagrams for $\langle x_{3,4} x_{3,4} \rangle$

We evaluate the diagrams (6.19)–(6.24) and (6.26)–(6.31) by taking the differences between (6.4)–(6.8) and (6.12)–(6.16) with help of the theorems.

The difference between (6.19) and (6.4) reads

$$\Delta \mathcal{A}^{4\mathcal{CC}} \equiv \mathcal{A}_{7,7}^{4\mathcal{CC}} - \mathcal{A}_{i,i}^{4\mathcal{CC}} \\ = \delta_{J\bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3} \right)^2 \sum_{J',J''=0}^{2j} n(2J' + 1)(2J'' + 1) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ \times \int^{\bar{\Lambda}_p} \frac{d^2 \widetilde{q}}{(2\pi)^2} \frac{\frac{4}{9} \widetilde{q}^2}{M_{3,0}(\widetilde{A}(J''), \widetilde{B}(J''), \widetilde{C}(J''); \widetilde{q})}, \quad (6.81)$$

whose UV part vanishes in the Moyal limit from the same reason why the last term in (6.54) vanishes. From Theorem 3, the UV part of the non-planar counterpart does not contribute. It is easy to see that the IR parts of (6.81) and the corresponding expression

for the non-planar diagrams (the contribution from $J'' = 0$) both vanish in the limit. Therefore, (6.19) and (6.26) coincide to (6.4) and (6.12) in the Moyal limit, respectively.

Let us next evaluate the difference between (6.21) and (6.6):

$$\begin{aligned}
\Delta \mathcal{A}^{3\mathcal{DD}} &\equiv \mathcal{A}_{7,7}^{3\mathcal{DD}} - \mathcal{A}_{i,i}^{3\mathcal{DD}} \\
&= \delta_{J\bar{J}} \delta_{m-\bar{m}} \left(\frac{M}{3}\right)^2 \left[\sum_{J'=0}^{2j} \sum_{J''=1}^{2j} \frac{(2J'+1)(2J''+1)}{J''(J''+1)} (J(J+1) - J'(J'+1))^2 \right. \\
&\quad \times n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \int \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{\frac{4}{9} \tilde{q}^2}{M_{3,1} \left(\tilde{A}(J'), \tilde{B}(J'), \tilde{C}(J'); \tilde{D}(J''); \tilde{q}, \tilde{p} \right)} \\
&\quad + \sum_{J'=0}^{2j} \sum_{J''=0}^{2j} \frac{2J'+1}{J''+1} (J'+J''+J+2)(J'-J''+J)(-J'+J''+J+1)(J'+J''-J+1) \\
&\quad \times n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \int \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{\frac{4}{9} \tilde{q}^2}{M_{3,1} \left(\tilde{A}(J'), \tilde{B}(J'), \tilde{C}(J'); \tilde{E}(J''); \tilde{q}, \tilde{p} \right)} \\
&\quad + \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} \frac{2J'+1}{J''} (J'+J''+J+1)(J'-J''+J+1)(-J'+J''+J)(J'+J''-J) \\
&\quad \times n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \int \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{\frac{4}{9} \tilde{q}^2}{M_{3,1} \left(\tilde{A}(J'), \tilde{B}(J'), \tilde{C}(J'); \tilde{F}(J''); \tilde{q}, \tilde{p} \right)} \left. \right]. \quad (6.82)
\end{aligned}$$

We read off the parameters in the theorems for each of the terms that satisfy $w \leq 0$, $W^- \geq 2$, $D_0 \geq 1$, and $D_1 \geq 2$, meaning that the UV parts of $\mathcal{A}_{7,7}^{3\mathcal{DD}}$ and $\mathcal{B}_{7,7}^{3\mathcal{DD}}$ coincide with those of $\mathcal{A}_{i,i}^{3\mathcal{DD}}$ and $\mathcal{B}_{i,i}^{3\mathcal{DD}}$ in the Moyal limit, respectively. In addition, the IR parts of (6.82) and the corresponding non-planar contributions both behave as $(M/3)^2 \times \mathcal{O}(\delta^0)$, which is irrelevant. Therefore, we can say that $\mathcal{A}_{7,7}^{3\mathcal{DD}}$ and $\mathcal{B}_{7,7}^{3\mathcal{DD}}$ has the same contribution as $\mathcal{A}_{i,i}^{3\mathcal{DD}}$ and $\mathcal{B}_{i,i}^{3\mathcal{DD}}$ in the limit, respectively.

By repeating the same manipulation, the quantities (6.19)–(6.24) and (6.26)–(6.31) coincide with (6.4)–(6.8) and (6.12)–(6.16), respectively. Furthermore, we can show that the residual diagrams (6.9), (6.10), (6.17) and (6.18) for $\langle x_{3,4} x_{3,4} \rangle$ vanish in the same way.

Combining the results obtained above, we see that the amplitudes $\mathcal{A}_{i,\bar{i}}(p; Jm; \bar{J}\bar{m})$ and $\mathcal{B}_{i,\bar{i}}(p; Jm; \bar{J}\bar{m})$ ($i, \bar{i} = 1, \dots, 7$) in (6.1) vanish for $i \neq \bar{i}$ and those for $i = \bar{i}$ become independent of the value of i in the Moyal limit.

7 Amplitudes and effective action

In this section, we sum up the contributions from the various diagrams obtained in the previous section, and evaluate the summations with respect to J' and J'' to obtain the one-loop effective action for the scalar kinetic terms in the successive limits (the Moyal limit and the commutative limit).

7.1 Planar diagrams

The UV part of the planar contributions is given by the summation of (6.56), (6.65), (6.69)–(6.71):

$$\mathcal{A}_{i,i}^{\text{UV}} \equiv \mathcal{A}_{i,i}^{\text{UV, single}} + \mathcal{A}_{i,i}^{\text{UV, double}} + \dots \quad (7.1)$$

with

$$\mathcal{A}_{i,i}^{\text{UV, single}} \equiv \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \sum_{J'=1}^{2j} \left\{ -\ln(J' + 1/3) + \ln(J' + 2/3) - \ln(J' + 1) + \ln J' \right\}, \quad (7.2)$$

$$\begin{aligned} \mathcal{A}_{i,i}^{\text{UV, double}} \equiv & \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ & \times \left[4J'J'' (\tilde{p}^2 + J^2) \left\{ \hat{L}(\tilde{A}(J'), \tilde{B}(J''); \tilde{p}) + \hat{L}(\tilde{A}(J'), \tilde{C}(J''); \tilde{p}) \right\} \right. \\ & + \left\{ 4J'J''\tilde{p}^2 + \frac{2J'}{J''} \left[(J'^2 - J''^2)^2 - 2J^2J'^2 + J^4 \right] \right\} \\ & \left. \times \left\{ -2\hat{L}(\tilde{A}(J'), \tilde{D}(J''); \tilde{p}) + \hat{L}(\tilde{A}(J'), \tilde{E}(J''); \tilde{p}) + \hat{L}(\tilde{A}(J'), \tilde{F}(J''); \tilde{p}) \right\} \right], \end{aligned} \quad (7.3)$$

and the IR part is given by the summation of (6.57), (6.66), (6.72)–(6.74):

$$\mathcal{A}_{i,i}^{\text{IR}} \equiv \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \frac{\tilde{p}^2 - \tilde{A}(J)}{\tilde{p}^2 + \tilde{A}(J)} \ln \delta + \dots \quad (7.4)$$

Note that the UV divergences containing $\ln \tilde{\Lambda}_p$ are canceled in the sums and do not appear in either (7.3) or (7.4), which is expected from the supersymmetry and supports the softness of the mass M . We easily see that (7.2) and (7.4) vanish in the Moyal limit (6.32) (followed by $\delta \rightarrow 0$ for the IR part) :

$$\mathcal{A}_{i,i}^{\text{UV, single}} \rightarrow 0, \quad \mathcal{A}_{i,i}^{\text{IR}} \rightarrow 0, \quad (7.5)$$

because (7.2) behaves as $\mathcal{O}(\frac{1}{n} \ln n) \rightarrow 0$ as $n = 2j + 1 \rightarrow \infty$. Therefore only (7.3) has a sensible contribution in the Moyal limit. In the following, we separately evaluate the contributions of (7.3) from Region I and Region II:

$$\mathcal{A}_{i,i}^{\text{UV, double}} = \mathcal{A}_{i,i}^{\text{UV, double, I}} + \mathcal{A}_{i,i}^{\text{UV, double, II}}. \quad (7.6)$$

7.1.1 $\mathcal{A}_{i,i}^{\text{UV, double, I}}$

In terms of the rescaled variables

$$u = \frac{M}{3} J, \quad u' = \frac{M}{3} J', \quad u'' = \frac{M}{3} J'', \quad (7.7)$$

the $6j$ symbol can be approximated as (D.23). Also,

$$\tilde{A}(J') \simeq \left(\frac{M}{3}\right)^2 (u')^2, \quad \tilde{B}(J'') \simeq \tilde{C}(J'') \simeq \tilde{D}(J'') \simeq \tilde{E}(J'') \simeq \tilde{F}(J'') \simeq \left(\frac{M}{3}\right)^2 (u'')^2, \quad (7.8)$$

and all the \hat{L} functions appearing in (7.3) have the same leading-order behavior:

$$\begin{aligned} \hat{L}(\tilde{P}(J'), \tilde{Q}(J''); \tilde{p}^2) &\simeq \left(\frac{M}{3}\right)^2 \frac{1}{\sqrt{[p^2 + (u' + u'')^2] [p^2 + (u' - u'')^2]}} \\ &\times \ln \left(\frac{p^2 + (u')^2 + (u'')^2 + \sqrt{[p^2 + (u' + u'')^2] [p^2 + (u' - u'')^2]}}{p^2 + (u')^2 + (u'')^2 - \sqrt{[p^2 + (u' + u'')^2] [p^2 + (u' - u'')^2]}} \right). \end{aligned} \quad (7.9)$$

Recall that the external momentum $|p|$ is assumed to be the same order as u . In this region, the summation can be approximated by the integral

$$\left(\frac{M}{3}\right)^2 \sum_{J', J''} \simeq \frac{1}{2} \int \int_{u \leq u' + u'' \leq u_B, -u \leq u' - u'' \leq u} du' du'', \quad (7.10)$$

where ⁸

$$u_B = \frac{M}{3} J_B = \mathcal{O}((1/M)^{2\alpha-1}) \gg 1 \quad (7.11)$$

and the prefactor $\frac{1}{2}$ reflects the fact that only the cases of $J' + J'' + J$ being even contribute to the summation.

Then the double sum part in Region I can be expressed as

$$\mathcal{A}_{i,i}^{\text{UV, double, I}} \simeq \delta_{J \bar{J}} \delta_{m - \bar{m}} \frac{2}{\pi^2} (p^2 + u^2) \int \int_{u \leq u' + u'' \leq u_B, -u \leq u' - u'' \leq u} du' du''$$

⁸Recall that $1/2 < \alpha < 1$.

$$\begin{aligned}
& \times \frac{u'u''}{\sqrt{[p^2 + (u' + u'')^2][p^2 + (u' - u'')^2][(u' + u'')^2 - u^2][u^2 - (u' - u'')^2]}} \\
& \times \ln \left(\frac{p^2 + (u')^2 + (u'')^2 + \sqrt{[p^2 + (u' + u'')^2][p^2 + (u' - u'')^2]}}{p^2 + (u')^2 + (u'')^2 - \sqrt{[p^2 + (u' + u'')^2][p^2 + (u' - u'')^2]}} \right) \\
& = \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{\pi^2} (p^2 + u^2) \int_1^{\tilde{\Lambda}} dz \int_a^1 dw f(z, w),
\end{aligned} \tag{7.12}$$

where the integration variables have been changed from (u', u'') to (z, w) by

$$u' + u'' = \sqrt{(p^2 + u^2)z^2 - p^2}, \quad |u' - u''| = \sqrt{(p^2 + u^2)w^2 - p^2}, \tag{7.13}$$

and we have defined

$$\tilde{\Lambda} \equiv \sqrt{\frac{p^2 + u_B^2}{p^2 + u^2}} \simeq \frac{u_B}{\sqrt{p^2 + u^2}}, \quad a \equiv \sqrt{\frac{p^2}{p^2 + u^2}}, \tag{7.14}$$

$$f(z, w) \equiv \frac{z^2 - w^2}{\sqrt{(z^2 - 1)(z^2 - a^2)(1 - w^2)(w^2 - a^2)}} \ln \left(\frac{z + w}{z - w} \right). \tag{7.15}$$

The leading contribution of the integration (7.12) comes from the region $z \sim \infty$ where the integrand behaves as

$$f(z, w) \sim \frac{2w}{\sqrt{(1 - w^2)(w^2 - a^2)}} \frac{1}{z} \equiv f_0(z, w), \tag{7.16}$$

which gives a singular behavior upon the integration:

$$\int_1^{\tilde{\Lambda}} dz \int_a^1 dw f_0(z, w) = \int_1^{\tilde{\Lambda}} dz \frac{\pi}{z} = \pi \ln \tilde{\Lambda} \simeq \pi \ln \left(\frac{u_B}{\sqrt{p^2 + u^2}} \right). \tag{7.17}$$

For the quantity subtracted by the singular part,

$$I(a) \equiv \lim_{\tilde{\Lambda} \rightarrow \infty} \int_1^{\tilde{\Lambda}} dz \int_a^1 dw [f(z, w) - f_0(z, w)], \tag{7.18}$$

we have analytically computed both $I(0)$ and $\lim_{a \rightarrow 1-0} I(a)$ to provide the identical result $\pi(-\ln 2 + 1)$. Furthermore, numerical computations for general $0 < a < 1$ (from, e.g., Mathematica) strongly suggest that $I(a)$ is indeed a constant independent of a :

$$I(a) = \pi(-\ln 2 + 1) \quad \text{for} \quad 0 \leq a < 1. \tag{7.19}$$

We proceed assuming that this is correct.

Combining the above results, we eventually have

$$\mathcal{A}_{i,i}^{\text{UV, double, I}} \simeq \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{\pi} (p^2 + u^2) \left(\ln u_B - \frac{1}{2} \ln(p^2 + u^2) - \ln 2 + 1 \right) \tag{7.20}$$

in the Moyal limit.

7.1.2 $\mathcal{A}_{i,i}^{\text{UV, double, II}}$

In Region II, J' and J'' satisfy

$$J', J'' \geq \frac{J_B - J}{2}, \quad J \geq |\Delta|. \quad (\Delta \equiv J' - J'') \quad (7.21)$$

Recalling that $J_B = \mathcal{O}(j^\alpha)$ and $J = \mathcal{O}(j^{1/2})$, we see

$$j, J', J'' \gg J, |\Delta| \quad \text{for} \quad j \gg 1. \quad (7.22)$$

In this region, the summation of J' and J'' cannot be evaluated by integrals as we have done in Region I. As seen in appendix D.1, the $6j$ symbol $\left\{ \begin{smallmatrix} J' & J'' & J \\ j & j & j \end{smallmatrix} \right\}$ in this region can be well approximated by using Edmonds' formula (D.6), which leads to

$$\begin{aligned} \left\{ \begin{smallmatrix} J' & J'' & J \\ j & j & j \end{smallmatrix} \right\}^2 &\simeq \frac{1}{n} \frac{1}{2J'' + 1} (d_{0\Delta}^J(\beta))^2, \\ \cos \beta &= \frac{1}{2} \sqrt{\frac{J''(J'' + 1)}{j(j + 1)}} = \frac{J_+}{4j} \left[1 + \mathcal{O}\left(\frac{\Delta}{J_+}, \frac{1}{J''}\right) \right] \quad \text{for} \quad 0 < \beta < \frac{\pi}{2}, \end{aligned} \quad (7.23)$$

where $J_+ \equiv J' + J''$ and $d_{MM'}^J(\beta)$ is a real function related to the Wigner D -function (see (D.8)). Since p^2 is assumed to be of the same order as $u^2 = \mathcal{O}(M^0)$, we can also see that all the functions \hat{L} appearing in $\mathcal{A}_{i,i}^{\text{UV, double, II}}$ commonly behave as

$$\hat{L}(\tilde{P}(J'), \tilde{Q}(J''); \tilde{p}^2) \simeq \frac{4}{J_+^2} \left(1 + \mathcal{O}\left(\frac{\Delta}{J_+}\right) \right). \quad (7.24)$$

Together with the above approximations, it turns out that (7.3) in Region II takes a simple form:

$$\mathcal{A}_{i,i}^{\text{UV, double, II}} \simeq \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{2}{\pi} (p^2 + u^2 + \mathcal{O}(M)) \sum_{\Delta=-J}^J \sum_{J_B \leq J_+ \leq 4j}^{(2\text{-step})} \frac{(d_{0\Delta}^J(\beta))^2}{J_+} \left(1 + \mathcal{O}\left(\frac{\Delta}{J_+}\right) \right). \quad (7.25)$$

Note that no Δ -dependence remains in the leading term except for the d -function.

The variable J_+ in the sum runs by two steps as signified, because J_+ must take even (odd) integers for a fixed Δ being even (odd). Thus the summation separates into those over even integers for both Δ and J_+ and odd integers for both Δ and J_+ . Let us consider replacing the latter summation over odd Δ and odd J_+ with the summation over odd Δ and even J_+ by increasing or reducing the value of J_+ by one. Since the error induced by

this replacement is of the order $\mathcal{O}(J_+^{-2})$, we can rewrite the summation in (7.25) as

$$\sum_{\Delta=-J}^J \sum_{J_B \leq J_+ \leq 4j}^{(2\text{-step})} \frac{(d_{0\Delta}^J(\beta))^2}{J_+} \left(1 + \mathcal{O}\left(\frac{\Delta}{J_+}\right)\right) = \sum_{J_B \leq J_+ \leq 4j}^{(J_+:\text{even})} \frac{1}{J_+} \sum_{\Delta=-J}^J (d_{0\Delta}^J(\beta))^2 + \mathcal{R}(J), \quad (7.26)$$

where

$$\mathcal{R}(J) \equiv \sum_{J_B \leq J_+ \leq 4j}^{(J_+:\text{even})} \mathcal{O}(J_+^{-2}) \sum_{-J \leq \Delta \leq J}^{(\Delta:\text{odd})} (d_{0\Delta}^J(\beta))^2. \quad (7.27)$$

By using the identity $\sum_{\Delta=-J}^J (d_{0\Delta}^J(\beta))^2 = 1$ (see (D.14)), we see that $\mathcal{R}(J)$ does not contribute in the Moyal limit as

$$|\mathcal{R}(J)| \leq \left| \sum_{J_B \leq J_+ \leq 4j}^{(J_+:\text{even})} \mathcal{O}(J_+^{-2}) \right| \leq (\text{const.}) \left(\frac{1}{J_B} - \frac{1}{4j} \right) \rightarrow 0. \quad (7.28)$$

Thus we end up with a simple sum to be evaluated as

$$\sum_{J_B \leq J_+ \leq 4j}^{(J_+:\text{even})} \frac{1}{J_+} = \frac{1}{2} \int_{\frac{J_B}{4j}}^1 \frac{dX}{X} = \frac{1}{2} \ln \frac{4j}{J_B} = \frac{1}{2} \ln \frac{2\Lambda_j}{u_B} \quad (7.29)$$

with $\Lambda_j \equiv \frac{M}{3} \cdot 2j$, and the amplitude becomes

$$\mathcal{A}_{i,i}^{\text{UV, double, II}} \simeq \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{\pi} (p^2 + u^2) \ln \frac{2\Lambda_j}{u_B}. \quad (7.30)$$

7.1.3 Total contribution from the planar diagrams in the Moyal limit

Combining the results (7.5), (7.20), and (7.30), we obtain the Moyal limit of the total contribution from the planar diagrams:

$$\begin{aligned} \mathcal{A}_{i,i} &= \mathcal{A}_{i,i}^{\text{IR}} + \mathcal{A}_{i,i}^{\text{UV}} = \mathcal{A}_{i,i}^{\text{UV, double, I}} + \mathcal{A}_{i,i}^{\text{UV, double, II}} + \dots \\ &\simeq \delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{\pi} (p^2 + u^2) \left(\ln \Lambda_j - \frac{1}{2} \ln(p^2 + u^2) + 1 \right). \end{aligned} \quad (7.31)$$

The dependence on u_B cancels between the contributions from Region I and Region II as it should. The amplitude depends on the external momenta in the 2D plane p and in the decompactified fuzzy sphere u only through the combination $p^2 + u^2$, which suggests the restoration of 4D rotational symmetry from $\mathbb{R}^2 \times (\text{fuzzy } S^2)$ in the Moyal limit.

7.2 Non-planar diagrams

In the non-planar diagrams, the UV contributions (6.60), (6.67), (6.75)–(6.77) are summed as

$$\mathcal{B}_{i,i}^{\text{UV}} \equiv \mathcal{B}_{i,i}^{\text{UV, single}} + \mathcal{B}_{i,i}^{\text{UV, double}} + \dots \quad (7.32)$$

with

$$\begin{aligned} \mathcal{B}_{i,i}^{\text{UV, single}} \equiv & -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} (-1)^{J'+J''+J} n(2J'+1) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ & \times \left\{ -\ln(J''+1/3) + \ln(J''+2/3) - \ln(J''+1) + \ln J'' \right\}, \end{aligned} \quad (7.33)$$

$$\begin{aligned} \mathcal{B}_{i,i}^{\text{UV, double}} \equiv & -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \sum_{J'=0}^{2j} \sum_{J''=1}^{2j} (-1)^{J'+J''+J} n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \\ & \times \left[4J'J''(\tilde{p}^2 + J^2) \left\{ \widehat{L}(\tilde{A}(J'), \tilde{B}(J''); \tilde{p}) + \widehat{L}(\tilde{A}(J'), \tilde{C}(J''); \tilde{p}) \right\} \right. \\ & + \left\{ 4J'J''\tilde{p}^2 + \frac{2J'}{J''} \left[(J'^2 - J''^2)^2 - 2J^2J'^2 + J^4 \right] \right\} \\ & \left. \times \left\{ -2\widehat{L}(\tilde{A}(J'), \tilde{D}(J''); \tilde{p}) + \widehat{L}(\tilde{A}(J'), \tilde{E}(J''); \tilde{p}) + \widehat{L}(\tilde{A}(J'), \tilde{F}(J''); \tilde{p}) \right\} \right], \end{aligned} \quad (7.34)$$

and the IR contribution is given by the summation of (6.61), (6.68), (6.78)–(6.80):

$$\mathcal{B}_{i,i}^{\text{IR}} \equiv -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{4\pi} \left(\frac{M}{3}\right)^2 \frac{\tilde{p}^2 - \tilde{A}(J)}{\tilde{p}^2 + \tilde{A}(J)} \ln \delta + \dots \quad (7.35)$$

We see that cancellation of UV singular terms of $\ln \tilde{\Lambda}_p$ also occurs in the non-planar contributions.

Similarly to the planar case, (7.33) and (7.35) vanish in the Moyal limit,

$$\mathcal{B}_{i,i}^{\text{UV, single}} \rightarrow 0, \quad \mathcal{B}_{i,i}^{\text{IR}} \rightarrow 0, \quad (7.36)$$

and the remaining terms (7.34) are computed separately in Region I and Region II:

$$\mathcal{B}_{i,i}^{\text{UV, double}} = \mathcal{B}_{i,i}^{\text{UV, double, I}} + \mathcal{B}_{i,i}^{\text{UV, double, II}}. \quad (7.37)$$

We can evaluate $\mathcal{B}_{i,i}^{\text{UV, double, I}}$ and $\mathcal{B}_{i,i}^{\text{UV, double, II}}$ by using almost the same manipulations to derive (7.31). The only difference from the planar counterparts $\mathcal{A}_{i,i}^{\text{UV, double, I}}$ and $\mathcal{A}_{i,i}^{\text{UV, double, II}}$ is the existence of the sign factor $-(-1)^{J'+J''+J}$ in the summation.

In Region I, the $6j$ symbol is negligible unless $(-1)^{J'+J''+J} = 1$ from (D.22). Hence we can repeat the same calculation as in the planar case (section 7.1.1) to evaluate $\mathcal{B}_{i,i}^{\text{UV, double, I}}$, which leads to

$$\mathcal{B}_{i,i}^{\text{UV, double, I}} \simeq -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{\pi} (p^2 + u^2) \left(\ln u_B - \frac{1}{2} \ln (p^2 + u^2) - \ln 2 + 1 \right). \quad (7.38)$$

In Region II, the expression reduces to

$$\mathcal{B}_{i,i}^{\text{UV, double, II}} \simeq -\delta_{J\bar{J}} \delta_{m-\bar{m}} (-1)^J \frac{2}{\pi} (p^2 + u^2 + \mathcal{O}(M)) \sum_{J_B \leq J_+ \leq 4j}^{(J_+ : \text{even})} \frac{1}{J_+} \sum_{\Delta=-J}^J (-1)^\Delta (d_{0\Delta}^J(\beta))^2. \quad (7.39)$$

Combining (D.15) and the formula ⁹

$$\sum_{M''=-J}^J d_{MM''}^J(\beta_1) d_{M''M'}^J(\beta_2) = d_{MM'}^J(\beta_1 + \beta_2) \quad \text{for} \quad \beta_1 + \beta_2 \leq \pi, \quad (7.40)$$

we obtain

$$\sum_{\Delta=-J}^J (-1)^\Delta (d_{0\Delta}^J(\beta))^2 = d_{00}^J(2\beta). \quad (7.41)$$

Equation (D.8) gives the explicit form of the function $d_{00}^J(2\beta)$ as

$$\begin{aligned} d_{00}^J(2\beta) &= \frac{(-1)^J}{2^J J!} \frac{d^J}{dY^J} \{ (1-Y)^J (1+Y)^J \} \\ &= \frac{(-1)^J}{2^J} \sum_{r=0}^J (-1)^r \binom{J}{r}^2 (1+Y)^r (1-Y)^{J-r} \\ &= (-1)^J \left[1 + \sum_{r=1}^J (-1)^r \binom{J}{r} X^{2r} + \sum_{r=1}^J (-1)^r \binom{J}{r}^2 X^{2r} (1-X^2)^{J-r} \right], \end{aligned} \quad (7.42)$$

where $Y \equiv \cos(2\beta)$ and $X \equiv \cos \beta$. On the other hand, the summation $\sum_{J_B \leq J_+ \leq 4j}^{(J_+ : \text{even})} \frac{d_{00}^J(2\beta)}{J_+}$ can be converted into an integral as

$$\sum_{J_B \leq J_+ \leq 4j}^{(J_+ : \text{even})} \frac{d_{00}^J(2\beta)}{J_+} \rightarrow \frac{1}{2} \int_{\frac{J_B}{4j}}^1 \frac{dX}{X} d_{00}^J(2\beta). \quad (7.43)$$

Then we can evaluate the summation in (7.39) as

$$-(-1)^J \sum_{J_B \leq J_+ \leq 4j}^{(J_+ : \text{even})} \frac{d_{00}^J(2\beta)}{J_+} = -\frac{1}{2} \ln \frac{2\Lambda_j}{u_B} + \frac{1}{2} H_J, \quad (7.44)$$

⁹See, e.g., eq. (7) in Chapter 4.7.2 of [58].

where H_J denotes the harmonic number

$$H_J \equiv \sum_{n=1}^J \frac{1}{n} = \gamma + \psi(J+1) = - \sum_{r=1}^J \frac{(-1)^r}{r} \binom{J}{r} \quad (7.45)$$

and γ is the Euler constant. In particular, H_J is evaluated as

$$H_J \simeq \ln \left(\frac{3}{M} u \right) + \gamma + \mathcal{O}(M) \quad (7.46)$$

for small M . Plugging these results into (7.39), we obtain the non-planar contribution in Region II:

$$\mathcal{B}_{i,i}^{\text{UV, double, II}} \simeq -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{\pi} (p^2 + u^2) \left(\ln \frac{2\Lambda_j}{u_B} - \ln \left(\frac{3}{M} u \right) - \gamma \right). \quad (7.47)$$

Combining (7.36), (7.38) and (7.47), the Moyal limit of the total contribution from the non-planar diagrams becomes

$$\begin{aligned} \mathcal{B}_{i,i} &= \mathcal{B}_{i,i}^{\text{IR}} + \mathcal{B}_{i,i}^{\text{UV}} = \mathcal{B}_{i,i}^{\text{UV, double, I}} + \mathcal{B}_{i,i}^{\text{UV, double, II}} + \dots \\ &\simeq -\delta_{J\bar{J}} \delta_{m-\bar{m}} \frac{1}{\pi} (p^2 + u^2) \left(\ln \Lambda_j - \frac{1}{2} \ln(p^2 + u^2) + 1 - \ln \left(\frac{3}{M} u \right) - \gamma \right). \end{aligned} \quad (7.48)$$

The last two terms $-\ln \left(\frac{3}{M} u \right) - \gamma$ in the non-planar amplitude have no counterpart in the planar contribution (7.31). They arise from the asymptotic behavior of the harmonic number (7.46) that has been recognized as a “noncommutative anomaly” in scalar field theory on fuzzy S^2 [56]. Although this anomaly is finite in the theory on the fuzzy S^2 since the external angular momentum J is finite, it becomes singular in the Moyal limit; i.e., the large radius limit of the fuzzy sphere to the Moyal plane \mathbb{R}_Θ^2 . Actually, in (7.48), $3/M$ is nothing but the radius of the fuzzy S^2 which diverges in sending $M \rightarrow 0$ with fixed u . Note that the terms are expressed as $-\ln \left(\sqrt{\Theta} u \right) - \frac{1}{2} \ln \frac{u}{2} - \gamma$, in which the first term signifies the UV/IR mixing phenomenon [55]. Due to their u -dependence, the noncommutative anomaly in the non-planar amplitude prevents restoration of the 4D rotational symmetry, which makes a contrast to the planar case.

7.3 Moyal limit of the modes

In order to obtain the one-loop effective action in the Moyal limit, we have to know the concrete form of mapping from the modes $x_{i,Jm}(p)$ in the expansion by the fuzzy spherical harmonics (3.14) to the modes $x_i(p, p')$ expanded by plane waves on the Moyal plane,

$$X_i(x) = \int \frac{d^2 p}{(2\pi)^2} e^{ip \cdot x} \int \frac{d^2 p'}{(2\pi)^2} e^{ip' \cdot \hat{x}} \otimes x_i(p, p'), \quad (7.49)$$

where $\hat{x} = (\hat{\xi}, \hat{\eta})$ are the coordinates of the Moyal plane (\mathbb{R}_Θ^2) satisfying

$$[\hat{\xi}, \hat{\eta}] = i\Theta, \quad (7.50)$$

and $p' \cdot \hat{x} = p'_1 \hat{\xi} + p'_2 \hat{\eta}$. When $n (= 2j + 1)$ is large, $x_{i,Jm}(p)$ can be expressed by $x_i(p, p')$:

$$x_{i,Jm}(p) = \int \frac{d^2 p'}{(2\pi)^2} \frac{1}{n} \text{tr}_n \left(\hat{Y}_{Jm}^{(jj)\dagger} e^{ip' \cdot \hat{x}} \right) x_i(p, p'). \quad (7.51)$$

Since we are eventually interested in the commutative limit $\Theta \rightarrow 0$, let us first consider the Moyal limit with Θ being small and $j \gg 1$. According to (E.29) and (E.84) in appendix E, the fuzzy spherical harmonics $\hat{Y}_{Jm}^{(jj)}$ can be approximated as

$$\hat{Y}_{Jm}^{(jj)} \simeq \begin{cases} \delta_{m0} \sqrt{2J+1} \mathbb{1}_n & \text{for } J \leq J_\varepsilon, \\ 2\pi(-i)^m \sqrt{2J} \int \frac{d^2 p'}{(2\pi)^2} \frac{e^{im\varphi_{p'}}}{|p'|} \delta(|p'| - u) e^{-ip' \cdot \hat{x}} & \text{for } J > J_\varepsilon \end{cases} \quad (7.52)$$

with J_ε being an integer of the order of $\mathcal{O}(M^{-\varepsilon})$ ($0 < \varepsilon \ll 1$). $\varphi_{p'}$ is a phase of the complex combination of the momentum: $p'_1 + ip'_2 = |p'| e^{i\varphi_{p'}}$. Plugging (7.52) into (7.51) leads to

$$x_{i,Jm}(p) = \begin{cases} \frac{1}{(2\pi\Theta)n} \sqrt{2J+1} \delta_{m0} x_i(p, 0) & \text{for } J \leq J_\varepsilon, \\ \frac{i^m \sqrt{2J}}{(2\pi\Theta)n} \int_0^{2\pi} \frac{d\varphi_{p'}}{2\pi} e^{im\varphi_{p'}} x_i(p, u e^{i\varphi_{p'}}) & \text{for } J > J_\varepsilon, \end{cases} \quad (7.53)$$

where we have used

$$\text{tr}_n (e^{ip \cdot \hat{x}} e^{iq \cdot \hat{x}}) = \frac{2\pi}{\Theta} \delta^2(p + q) \quad \text{for } n \sim \infty, \quad (7.54)$$

and the second argument of $x_i(p, u e^{i\varphi_{p'}})$ specifies the momentum in \mathbb{R}_Θ^2 in the form of the complex combination. In addition, we divide the summation over J into two parts:

$$\sum_{J=0}^{2j} = \sum_{J=0}^{J_\varepsilon} + \sum_{J=J_\varepsilon+1}^{2j}, \quad (7.55)$$

and transcribe the latter summation into the integral as

$$\sum_{J=J_\varepsilon+1}^{2j} \sum_{m=-J}^J \rightarrow \frac{3}{M} \int_0^\infty du \sum_{m \in \mathbb{Z}}. \quad (7.56)$$

7.4 Scalar kinetic terms in the one-loop effective action

Let us first consider rewriting the tree-level kinetic terms of the scalar fields in terms of the modes $x_i(p, p')$:

$$S_{2,B,x_i}^{\text{tree}} = \frac{n}{g_{2d}^2} \int \frac{d^2 p}{(2\pi)^2} \sum_{J=0}^{2j} \sum_{m=-J}^J (-1)^m \left(p^2 + \frac{2M^2}{81} + \frac{M^2}{9} J(J+1) \right) \times \text{tr}_k [x_{i,J-m}(-p) x_{i,J-m}(p)]. \quad (7.57)$$

In the Moyal limit, we replace the modes with (7.53) and take the limit of $M \rightarrow 0$ and $n \rightarrow \infty$ with fixing $\Theta = \frac{18}{M^2 n}$. It turns out that the contribution from $0 \leq J \leq J_\epsilon$ disappears, and the result reads

$$S_{2,B,x_i}^{\text{tree}} \rightarrow \frac{1}{g_{4d}^2} \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \text{tr}_k (x_i(-\mathbf{p}) x_i(\mathbf{p})) \mathbf{p}^2 \quad (7.58)$$

with the 4D coupling $g_{4d}^2 \equiv 2\pi\Theta g_{2d}^2$ and four-momentum $\mathbf{p} \equiv (p, p')$. We have also used $x_i(\mathbf{p}) = x_i(p, p')$, $\mathbf{p}^2 = p^2 + u^2$ and

$$\int \frac{d^4 \mathbf{p}}{(2\pi)^4} = \int \frac{d^2 p}{(2\pi)^2} \int_0^\infty du u \int_0^{2\pi} \frac{d\varphi'_p}{(2\pi)^2}. \quad (7.59)$$

This reproduces the tree-level kinetic terms of 4D scalar fields.

We repeat the same procedure for the one-loop part of the effective action for the operators $\text{tr}_k(x_{i,J-m}(-p) x_{i,J-m}(p))$ and $\text{tr}_k(x_{i,J-m}(-p)) \text{tr}_k(x_{i,J-m}(p))$ which is nothing but the negative of the scalar two-point function (6.1) with (7.31) and (7.48). Since all the fields were rescaled as (4.1) in the perturbative calculation, we rescale them back to the original expression. Then, the result in the Moyal limit becomes

$$S_{2,B,x_i}^{1\text{-loop}} \rightarrow \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \text{tr}_k [x_i(-\mathbf{p}) x_i(\mathbf{p})] \mathbf{p}^2 \times \frac{k}{4\pi^2} \left\{ \ln \Lambda_j - \frac{1}{2} \ln(\mathbf{p}^2) + 1 \right\} + \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \text{tr}_k (x_i(-\mathbf{p})) \text{tr}_k (x_i(\mathbf{p})) \mathbf{p}^2 \times \frac{-1}{4\pi^2} \left\{ \ln \Lambda_j - \frac{1}{2} \ln(\mathbf{p}^2) + 1 - \ln \left(\frac{3}{M} u \right) - \gamma \right\}. \quad (7.60)$$

We decompose the modes to the $SU(k)$ part and the overall $U(1)$ part:

$$x_i(\mathbf{p}) = x_i^{SU(k)}(\mathbf{p}) + x_i^{U(1)}(\mathbf{p}) \quad \text{with} \quad \text{tr}_k \left(x_i^{SU(k)}(\mathbf{p}) \right) = 0, \quad x_i^{U(1)}(\mathbf{p}) \propto \mathbb{1}_k, \quad (7.61)$$

and express the effective action up to the one-loop order ((7.58) + (7.60)):

$$\Gamma_{2,B,x_i}^{1\text{-loop}} \equiv S_{2,B,x_i}^{\text{tree}} + S_{2,B,x_i}^{1\text{-loop}}$$

$$\begin{aligned}
&= \frac{1}{g_{4d}^2} \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \frac{1}{k} \text{tr}_k \left(x_i^{U(1)}(-\mathbf{p}) \right) \text{tr}_k \left(x_i^{U(1)}(\mathbf{p}) \right) \mathbf{p}^2 \\
&\quad \times \left[1 + \frac{g_{4d}^2 k}{4\pi^2} \left\{ \ln \left(\frac{3}{M} u \right) + \gamma \right\} \right] \\
&+ \frac{1}{g_{4d}^2} \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \text{tr}_k \left[x_i^{SU(k)}(-\mathbf{p}) x_i^{SU(k)}(\mathbf{p}) \right] \mathbf{p}^2 \\
&\quad \times \left[1 + \frac{g_{4d}^2 k}{4\pi^2} \left\{ \ln \Lambda_j - \frac{1}{2} \ln(\mathbf{p}^2) + 1 \right\} \right]. \tag{7.62}
\end{aligned}$$

The wave function renormalization:

$$x_i^{SU(k)(R)}(\mathbf{p}) \equiv \left(1 + \frac{g_{4d}^2 k}{4\pi^2} \ln \frac{\Lambda_j}{\mu_R} \right)^{1/2} x_i^{SU(k)}(\mathbf{p}), \tag{7.63}$$

$$x_i^{U(1)(R)}(\mathbf{p}) \equiv \left(1 + \frac{g_{4d}^2 k}{4\pi^2} \ln \left(\frac{3}{M} e^\gamma \mu_R \right) \right)^{1/2} x_i^{U(1)}(\mathbf{p}) \tag{7.64}$$

(μ_R is the renormalization point) absorbs all the divergences arising in the Moyal limit and recasts the effective action as

$$\begin{aligned}
\Gamma_{2,B,x_i}^{1\text{-loop}} &= \frac{1}{g_{4d}^2} \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \frac{1}{k} \text{tr}_k \left(x_i^{U(1)(R)}(-\mathbf{p}) \right) \text{tr}_k \left(x_i^{U(1)(R)}(\mathbf{p}) \right) \mathbf{p}^2 + \Delta\Gamma \\
&+ \frac{1}{g_{4d}^2} \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \text{tr}_k \left[x_i^{SU(k)(R)}(-\mathbf{p}) x_i^{SU(k)(R)}(\mathbf{p}) \right] \mathbf{p}^2 \\
&\quad \times \left[1 + \frac{g_{4d}^2 k}{4\pi^2} \left\{ -\frac{1}{2} \ln \frac{\mathbf{p}^2}{\mu_R^2} + 1 \right\} + \mathcal{O}(g_{4d}^4) \right], \tag{7.65}
\end{aligned}$$

where

$$\Delta\Gamma \equiv \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \text{tr}_k \left(x_i^{U(1)(R)}(-\mathbf{p}) \right) \text{tr}_k \left(x_i^{U(1)(R)}(\mathbf{p}) \right) \frac{\mathbf{p}^2}{4\pi^2} \ln \frac{u}{\mu_R} + \mathcal{O}(g_{4d}^2). \tag{7.66}$$

At this stage, the limit of Step 2 (commutative limit $\Theta \rightarrow 0$) can be trivially taken to give the final result (7.65) and (7.66). The $U(1)$ part is not $SO(4)$ invariant due to the noncommutative anomaly in $\Delta\Gamma$ (recall that u is the momentum in the plane obtained from the fuzzy S^2), while the anomaly is harmless in the $SU(k)$ part at the one-loop level. Beyond the one-loop order, however, such $SO(4)$ breaking could also affect the $SU(k)$ sector in the kinetic terms. Here, the $U(1)$ part does not couple with the $SU(k)$ part in the quadratic terms of the effective action, which is the case for any quadratic term to all the orders for a group theoretical reason. We also expect that the interaction terms receive no radiative corrections except those absorbed by the wave function renormalization which is the same as the situation in the ordinary $\mathcal{N} = 4$ SYM on \mathbb{R}^4 [57, 59, 60]. For n -point amplitudes with $n \geq 3$, the UV divergence will be at most logarithmic from the power

counting and parity invariance in the Moyal limit. The leading divergence would be canceled by supersymmetry as seen in the two-point amplitudes, and the result would be UV finite. In such UV-finite amplitudes, there is no obstruction in the commutative limit on the convergence to the corresponding results in the ordinary $\mathcal{N} = 4$ SYM. As in the ordinary $\mathcal{N} = 4$ SYM [57], radiative corrections to the quadratic terms in the effective action would be gauge-dependent, and thus the noncommutative anomaly that appears accompanied by the wave function renormalization would be a gauge artifact not affecting gauge invariant observables. This is supported by the analysis of the 4D $\mathcal{N} = 4$ SYM theory in $\mathbb{R}^2 \times \mathbb{R}_\Theta^2$ in the light-cone gauge [54], which shows that the limit $\Theta \rightarrow 0$ is continuous to the ordinary theory defined on \mathbb{R}^4 to all the orders in perturbation theory; namely, the noncommutative anomaly does not appear.

8 Conclusion and discussion

Starting with the mass deformation of 2D $\mathcal{N} = (8, 8)$ $U(N)$ SYM, which preserves two supercharges, we have obtained 4D $\mathcal{N} = 4$ $U(k)$ SYM on $\mathbb{R}^2 \times (\text{fuzzy } S^2)$ around the fuzzy sphere classical solution of the 2D theory. The radius of the fuzzy S^2 is proportional to the inverse of the mass M and the noncommutativity Θ is proportional to the inverse of the product of $n = N/k$ and the mass squared M^2 . It is clear at the classical level that the two successive procedures, (1) decompactify the fuzzy S^2 to a noncommutative plane and (2) turn off the noncommutativity of the plane, derive the ordinary $\mathcal{N} = 4$ SYM on \mathbb{R}^4 . As a nontrivial check at the quantum level, we have computed the one-loop effective action with respect to the kinetic terms for scalar fields X_i ($i = 3, \dots, 7$), where the gauge is fixed to a Feynman-type gauge. The IR singularities turn out to be harmless in the above limits by introducing the IR cutoff δ at the intermediate step in the computation.

For the $SU(k)$ sector in the gauge group $U(k)$ of the 4D theory, which contains only the contribution of planar diagrams, the result coincides with the ordinary 4D SYM on \mathbb{R}^4 after the wave function renormalization. In particular, the $SO(4)$ rotational symmetry in \mathbb{R}^4 is not ruined by the quantum correction. On the other hand, the overall $U(1)$ sector including the contribution of non-planar diagrams has shown a “noncommutative anomaly”, which has no counterpart in the ordinary SYM. Due to this anomaly, the $SO(4)$ symmetry does not appear to be restored. Also, such an anomaly may affect the $SU(k)$ sector beyond the one-loop order. However, we expect that it arises only accompanied by the wave function renormalization. Since the wave function renormalization is gauge dependent as in the ordinary 4D $\mathcal{N} = 4$ SYM, the anomaly is expected to be also a gauge artifact and will not arise in computing gauge invariant observables as far as the

gauge symmetry is respected. Of course, it is desirable to calculate other kinetic terms and interaction terms of the one-loop effective action as well as higher-loop corrections in order to make the expectation firmer. Due to the technical complexity, we will leave this for future work. It will be important to confirm the harmlessness of the noncommutative anomaly by numerical simulations.

Rigorously speaking, the gauge invariant observables should be invariant under the gauge transformation of the theory on $\mathbb{R}^2 \times (\text{fuzzy } S^2)$ before taking the Moyal limit. They should be nonlocal in the fuzzy S^2 directions and have the angular momentum $J = 0$. Interestingly, however, for field variables in the effective action with $J \ll j$, we can consider local observables with nonzero J in the following reason. The scalar field $X_i(x)$, whose mode expansion is given by (3.14), transforms under the gauge transformation with the parameter $\Omega(x) = \int \frac{d^2 p}{(2\pi)^2} \tilde{\Omega}(p)$ and (C.27) as

$$\begin{aligned} \delta X_i(x) &= i[\Omega(x), X_i(x)] \\ &= i \int \frac{d^2 p_1}{(2\pi)^2} \int \frac{d^2 p_2}{(2\pi)^2} e^{i(p_1+p_2) \cdot x} \sum_{(J,m)} \sum_{(J_1,m_1)} \sum_{(J_2,m_2)} \hat{C}_{J_1 m_1(jj) J_2 m_2(jj)}^{J m(jj)} \\ &\quad \times \hat{Y}_{Jm}^{(jj)} \otimes [\omega_{J_1 m_1}(p_1) x_{i, J_2 m_2}(p_2) - (-1)^{J_1+J_2+J} x_{i, J_2 m_2}(p_2) \omega_{J_1 m_1}(p_1)] , \quad (8.1) \end{aligned}$$

where we have used (B.18) and (B.24) with the notation (6.3). The expression does not vanish by taking the partial trace tr_k , but does under the total trace $\text{Tr} = \text{tr}_n \text{tr}_k$. The total trace $\text{Tr } X_i(x)$ yields the observables $\text{tr}_k x_{i,00}(p)$. For the case of external angular momenta sufficiently smaller than the cutoff, i.e. $J_1, J_2 \ll j$ and thus $J \ll j$, the contribution of $\hat{C}_{J_1 m_1(jj) J_2 m_2(jj)}^{J m(jj)}$, which includes the $6j$ symbol as in (B.23), is greatly suppressed when $J_1 + J_2 + J$ is odd, according to the formula (D.21). This allows us to replace the sign factor $(-1)^{J_1+J_2+J}$ with unity. Hence, we can effectively consider the partial trace $\text{tr}_k X_i(x)$ or equivalently $\text{tr}_k x_{i,Jm}(p)$ as gauge invariant observables¹⁰. By repeating a similar argument, $\text{tr}_k X_i(x)^\ell$ ($\ell = 1, 2, \dots$) can be regarded as gauge invariant observables.

Since the overall $U(1)$ part is uninteresting in the target theory (the ordinary $\mathcal{N} = 4$ SYM on \mathbb{R}^4), it is better to consider the observables subtracted by that part. For example,

¹⁰ Nevertheless, the two-point function of $\text{tr}_k x_{i,Jm}(p)$ yields the $SO(4)$ breaking term $\Delta\Gamma$ in (7.65). There we used the ordinary renormalization prescription of local field theory, i.e., subtraction by local counter terms, which would not respect the full gauge invariance in noncommutative gauge theory. In fact, under the gauge transformation (8.1), the local counter term corresponding to (7.64) varies by the amount of the product of the divergent factor ($\ln M$) and the suppression for $J_1 + J_2 + J$ odd, which could be nonvanishing. We expect that finite quantities free from the renormalization factors such as (8.4) will avoid the issue and show restoration of the $SO(4)$ symmetry.

with use of the field variables

$$X_i^{(\text{TL})}(x) \equiv \sum_{J=0}^{2j} \sum_{m=-J}^J \int \frac{d^2 p}{(2\pi)^2} e^{ip \cdot x} \hat{Y}_{Jm}^{(jj)} \otimes x_{i,Jm}^{(\text{TL})}(p) \quad (8.2)$$

with

$$x_{i,Jm}^{(\text{TL})}(p) \equiv x_{i,Jm}(p) - \frac{1}{k} (\text{tr}_k x_{i,Jm}(p)) \mathbb{1}_k, \quad (8.3)$$

it would be worth seeing the restoration of the $SO(4)$ symmetry at the nonperturbative level by numerical simulation of the quantities ($L = 2, 3, \dots$):

$$\frac{\left\langle \prod_{\ell=1}^L \text{tr}_k \left(X_{i_\ell}^{(\text{TL})}(x_\ell)^2 \right) \right\rangle_{\text{conn.}}}{\left\langle \text{tr}_k \left(X_i^{(\text{TL})}(x)^2 \right) \right\rangle^L}. \quad (8.4)$$

The subscript “conn.” in the numerator means that the connected part of the L -point correlation function is taken. The denominator is introduced as having finite quantities independent of possible wave function renormalizations. The point x and the index $i \in \{3, \dots, 7\}$ in the denominator can be freely chosen as a reference.

Finally, we comment on the nonperturbative stability of the k -coincident fuzzy S^2 solution (3.1) with (3.3), which will be relevant in numerical simulation. As an illustration, tunneling amplitudes from the fuzzy S^2 solution

1. to the trivial vacuum ($X_a = 0$)
2. to the $(k-1)$ -coincident fuzzy S^2 solution

$$X_a(x) = \frac{M}{3} L'_a, \quad L'_a \equiv \begin{pmatrix} L_a^{(n)} \otimes \mathbb{1}_{k-1} & \\ & \mathbf{0}_n \end{pmatrix} \quad (8.5)$$

3. to the solutions

$$X_a(x) = \frac{M}{3} L''_a, \quad L''_a \equiv \begin{pmatrix} L_a^{(n)} \otimes \mathbb{1}_{k-2} & & \\ & L_a^{(n+\ell)} & \\ & & L_a^{(n-\ell)} \end{pmatrix} \quad (\ell = 1, 2, \dots, n-1) \quad (8.6)$$

4. to the solutions

$$X_a(x) = \frac{M}{3} L'''_a, \quad L'''_a \equiv \begin{pmatrix} L_a^{(n)} \otimes \mathbb{1}_{k-1} & & \\ & L_a^{(\ell)} & \\ & & L_a^{(n-\ell)} \end{pmatrix} \quad (\ell = 1, 2, \dots, n-1) \quad (8.7)$$

are evaluated in appendix F. Although the tunneling amplitudes are expected to be suppressed due to the infinite volume of the space-time \mathbb{R}^2 [46], we will see this in more detail. Indeed, by scaling the length of the spatial direction in the two-dimensional space-time faster than $1/M$, all the results are shown to become zero in the successive limits (Step 1 and Step 2 in section 4). This supports the nonperturbative stability of the solution (3.1) with (3.3) in taking the successive limits.

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A Deformed action in BTFT form

The deformed action in the BTFT description $S = S_b + S_f$ (S_b and S_f denote its bosonic and fermionic parts, respectively) is explicitly given by

$$\begin{aligned}
S_b = & \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \left\{ F_{12}^2 + (D_\mu B_{\mathbf{A}})^2 + (D_\mu X_{\underline{i}})^2 + \frac{1}{4} (D_\mu C)^2 + D_\mu \phi_+ D_\mu \phi_- \right. \\
& - [X_{\underline{i}}, B_{\mathbf{A}}]^2 - [X_3, X_4]^2 - [B_1, B_2]^2 - [B_2, B_3]^2 - [B_3, B_1]^2 \\
& + \frac{1}{4} [\phi_+, \phi_-]^2 + \frac{1}{4} [\phi_+, C] [C, \phi_-] - \frac{1}{4} [C, X_{\underline{i}}]^2 - \frac{1}{4} [C, B_{\mathbf{A}}]^2 \\
& + [\phi_+, X_{\underline{i}}] [X_{\underline{i}}, \phi_-] + [\phi_+, B_{\mathbf{A}}] [B_{\mathbf{A}}, \phi_-] \\
& + \frac{2M^2}{81} (B_{\mathbf{A}}^2 + X_{\underline{i}}^2) + \frac{M^2}{9} \left(\frac{C^2}{4} + \phi_+ \phi_- \right) - \frac{M}{2} C [\phi_+, \phi_-] \\
& \left. - \frac{4iM}{9} B_3 (F_{12} + i[X_3, X_4]) \right\}, \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
S_f = & \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \left\{ i\eta_+ D_\mu \psi_{-\mu} + i\eta_- D_\mu \psi_{+\mu} - \eta_+ [X_{\underline{i}}, \rho_{-\underline{i}}] - \eta_- [X_{\underline{i}}, \rho_{+\underline{i}}] \right. \\
& \left. + \psi_{+\mu} [\phi_-, \psi_{+\mu}] - \psi_{-\mu} [\phi_+, \psi_{-\mu}] + \psi_{+\mu} [C, \psi_{-\mu}] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \rho_{+\underline{i}} [\phi_{-}, \rho_{+\underline{i}}] - \rho_{-\underline{i}} [\phi_{+}, \rho_{-\underline{i}}] + \rho_{+\underline{i}} [C, \rho_{-\underline{i}}] \\
& + \chi_{+\mathbf{A}} [\phi_{-}, \chi_{+\mathbf{A}}] - \chi_{-\mathbf{A}} [\phi_{+}, \chi_{-\mathbf{A}}] + \chi_{+\mathbf{A}} [C, \chi_{-\mathbf{A}}] \\
& + \frac{1}{4} \eta_{+} [\phi_{-}, \eta_{+}] - \frac{1}{4} \eta_{-} [\phi_{+}, \eta_{-}] - \frac{1}{4} \eta_{+} [C, \eta_{-}] \\
& - \eta_{+} [B_{\mathbf{A}}, \chi_{-\mathbf{A}}] - \eta_{-} [B_{\mathbf{A}}, \chi_{+\mathbf{A}}] - 2\epsilon_{\mathbf{ABC}} \chi_{\mathbf{A}} [B_{\mathbf{B}}, \chi_{-\mathbf{C}}] \\
& + 2i\chi_{+1} (D_1\rho_{-3} + D_2\rho_{-4} - i[X_3, \psi_{-1}] - i[X_4, \psi_{-2}]) \\
& + 2i\chi_{+2} (D_1\rho_{-4} - D_2\rho_{-3} - i[X_4, \psi_{-1}] + i[X_4, \psi_{-2}]) \\
& + 2i\chi_{+3} (D_1\psi_{-2} - D_2\psi_{-1} + i[X_4, \rho_{-3}] - i[X_3, \rho_{-4}]) \\
& - 2i\chi_{-1} (D_1\rho_{+3} + D_2\rho_{+4} - i[X_3, \psi_{+1}] - i[X_4, \psi_{+2}]) \\
& - 2i\chi_{-2} (D_1\rho_{+4} - D_2\rho_{+3} - i[X_4, \psi_{+1}] + i[X_3, \psi_{+2}]) \\
& - 2i\chi_{-3} (D_1\psi_{+2} - D_2\psi_{+1} + i[X_4, \rho_{+3}] - i[X_3, \rho_{+4}]) \\
& + 2\psi_{+1} ([B_1, \rho_{-3}] + [B_2, \rho_{-4}] + [B_3, \psi_{-2}]) \\
& + 2\psi_{+2} ([B_1, \rho_{-4}] - [B_2, \rho_{-3}] - [B_3, \psi_{-1}]) \\
& + 2\rho_{+3} (-[B_1, \psi_{-1}] + [B_2, \psi_{-2}] - [B_3, \rho_{-4}]) \\
& + 2\rho_{+4} (-[B_1, \psi_{-2}] - [B_2, \psi_{-1}] + [B_3, \rho_{-3}]) \\
& + \frac{2M}{3} \psi_{+\mu} \psi_{-\mu} + \frac{2M}{9} \rho_{+\underline{i}} \rho_{-\underline{i}} + \frac{4M}{9} \chi_{+A} \chi_{-A} - \frac{M}{6} \eta_{+} \eta_{-} \Big\}. \tag{A.2}
\end{aligned}$$

B Fuzzy spherical harmonics

In this appendix, we give definitions and properties of various fuzzy spherical harmonics [61, 44] that are relevant in the text.

Let $|j\,r\rangle$ ($r = -j, -j+1, \dots, j$) an orthonormal basis of $n(= 2j+1)$ -dimensional space of a spin- j representation of $SU(2)$ normalized by

$$\langle j\,r | j'\,r' \rangle = \delta_{j\,j'} \delta_{r\,r'}. \tag{B.1}$$

Here j is assumed to take a non-negative integer or half-integer value. The $SU(2)$ generators L_a ($a = 1, 2, 3$) satisfying $[L_a, L_b] = i\epsilon_{abc} L_c$ act on the basis as

$$\begin{aligned}
L_3 |j\,r\rangle &= r |j\,r\rangle, \\
L_+ |j\,r\rangle &= \sqrt{(j-r)(j+r+1)} |j\,r+1\rangle, \\
L_- |j\,r\rangle &= \sqrt{(j+r)(j-r+1)} |j\,r-1\rangle,
\end{aligned} \tag{B.2}$$

where $L_{\pm} = L_1 \pm iL_2$.

By expressing $\{|j r\rangle\}$ as n -dimensional unit vectors as

$$|j - j\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad |j - j + 1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \quad |j j\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad (\text{B.3})$$

any $n \times n$ matrix M can be written as

$$M = \begin{pmatrix} M_{-j, -j} & M_{-j, -j+1} & \cdots & M_{-j, j} \\ M_{-j+1, -j} & M_{-j+1, -j+1} & \cdots & M_{-j+1, j} \\ \vdots & \vdots & & \vdots \\ M_{j, -j} & M_{j, -j+1} & \cdots & M_{j, j} \end{pmatrix} = \sum_{r, r'=-j}^j M_{r, r'} |j r\rangle \langle j r'|. \quad (\text{B.4})$$

The adjoint action of L_a to M is defined as

$$L_a \circ M \equiv [L_a, M] = \sum_{r, r'} M_{r, r'} \left(L_a |j r\rangle \langle j r'| - |j r\rangle \langle j r'| L_a \right). \quad (\text{B.5})$$

Then, it is easy to see that

$$[L_a \circ, L_b \circ] = i \epsilon_{abc} L_c \circ. \quad (\text{B.6})$$

B.1 (Scalar) fuzzy spherical harmonics

(Scalar) fuzzy spherical harmonics is defined by

$$\hat{Y}_{Jm}^{(jj)} \equiv \sqrt{n} \sum_{r, r'=-j}^j (-1)^{-j+r'} C_{j r j - r'}^{Jm} |j r\rangle \langle j r'|, \quad (\text{B.7})$$

where $C_{j r j - r'}^{Jm} \equiv \langle j j r - r' | J m \rangle$ is a Clebsch-Gordan (C-G) coefficient vanishing unless $m = r - r'$. In the basis (B.3), $\hat{Y}_{Jm}^{(jj)}$ is an $n \times n$ matrix whose (r, r') component is given by $\sqrt{n} (-1)^{-j+r'} C_{j r j - r'}^{Jm}$. Note that J and $m = r - r'$ take integer values as seen from the C-G coefficient.

From the definition (B.5),

$$L_3 \circ \hat{Y}_{Jm}^{(jj)} = m \hat{Y}_{Jm}^{(jj)}, \quad (\text{B.8})$$

and the recursion relation for C-G coefficients¹¹,

$$\sqrt{(c \pm \gamma)(c \mp \gamma + 1)} C_{a \alpha b \beta}^{c \gamma \mp 1}$$

¹¹See, e.g., eq. (4) in Chapter 8.6.2 of [58].

$$= \sqrt{(a \mp \alpha)(a \pm \alpha + 1)} C_{a\alpha \pm 1 b \beta}^{c\gamma} + \sqrt{(b \mp \beta)(b \pm \beta + 1)} C_{a\alpha b \beta \pm 1}^{c\gamma} \quad (\text{B.9})$$

leads to

$$\begin{aligned} L_+ \circ \hat{Y}_{Jm}^{(jj)} &= \sqrt{(J-m)(J+m+1)} \hat{Y}_{Jm+1}^{(jj)}, \\ L_- \circ \hat{Y}_{Jm}^{(jj)} &= \sqrt{(J+m)(J-m+1)} \hat{Y}_{Jm-1}^{(jj)}. \end{aligned} \quad (\text{B.10})$$

Therefore, we have

$$(L_a \circ)^2 \hat{Y}_{Jm}^{(jj)} = J(J+1) \hat{Y}_{Jm}^{(jj)}. \quad (\text{B.11})$$

$C_{j r j -r'}^{Jm}$ is real, and the relation

$$C_{j r j -r'}^{Jm} = C_{j r' j -r}^{J-m} \quad (\text{B.12})$$

obeys the identities [58]

$$C_{J_1 m_1 J_2 m_2}^{J_3 m_3} = (-1)^{J_1+J_2-J_3} C_{J_2 m_2 J_1 m_1}^{J_3 m_3} = (-1)^{J_1+J_2-J_3} C_{J_1 -m_1 J_2 -m_2}^{J_3 -m_3}. \quad (\text{B.13})$$

Then, the hermitian conjugate of $\hat{Y}_{Jm}^{(jj)}$ becomes

$$\left(\hat{Y}_{Jm}^{(jj)} \right)^\dagger = (-1)^m \hat{Y}_{J-m}^{(jj)}. \quad (\text{B.14})$$

For the n -dimensional trace “ tr_n ”, the orthonormality

$$\frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{Jm}^{(jj)} \right)^\dagger \hat{Y}_{J'm'}^{(jj)} \right\} = \delta_{JJ'} \delta_{mm'}, \quad (\text{B.15})$$

or equivalently

$$\frac{1}{n} \text{tr}_n \left\{ \hat{Y}_{Jm}^{(jj)} \hat{Y}_{J'm'}^{(jj)} \right\} = (-1)^m \delta_{JJ'} \delta_{m+m'0}, \quad (\text{B.16})$$

follows from the orthogonality of the C-G coefficients

$$\sum_{\alpha, \beta} C_{a\alpha b \beta}^{c\gamma} C_{a\alpha b \beta}^{c'\gamma'} = \delta_{cc'} \delta_{\gamma\gamma'}. \quad (\text{B.17})$$

Next, let us compute the trace of the product of three fuzzy spherical harmonics given by (3.18) in the text, which is equivalent to

$$\hat{Y}_{J_1 m_1}^{(jj)} \hat{Y}_{J_2 m_2}^{(jj)} = \sum_{J=0}^{2j} \sum_{m=-J}^J \hat{C}_{J_1 m_1 (jj) J_2 m_2 (jj)}^{Jm(jj)} \hat{Y}_{Jm}^{(jj)}. \quad (\text{B.18})$$

From the definition (B.7) and the identity ¹²

$$C_{J_1 m_1 J_2 m_2}^{J_3 m_3} = (-1)^{J_2+m_2} \sqrt{\frac{2J_3+1}{2J_1+1}} C_{J_2 -m_2 J_3 m_3}^{J_1 m_1}, \quad (\text{B.19})$$

¹²See, e.g., eq. (10) in Chapter 8.4.3 of [58].

we have

$$\hat{C}_{J_2 m_2(jj) J_3 m_3(jj)}^{J_1 m_1(jj)} = \sqrt{2J_2 + 1} \sum_{r, r', r''} C_{j r'' j r'}^{J_3 m_3} C_{j r j r'}^{J_1 m_1} C_{j r'' J_2 m_2}^{j r}. \quad (\text{B.20})$$

Furthermore,

$$\sum_{\alpha, \beta, \delta} C_{a \alpha b \beta}^{c \gamma} C_{d \delta b \beta}^{e \epsilon} C_{a \alpha f \varphi}^{d \delta} = (-1)^{b+c+d+f} \sqrt{(2c+1)(2d+1)} C_{c \gamma f \varphi}^{e \epsilon} \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix} \quad (\text{B.21})$$

(See, e.g., eq. (12) in Chapter 8.7.3 in [58]) and the property of the $6j$ symbol,

$$\begin{Bmatrix} a & b & C \\ A & B & c \end{Bmatrix} = \begin{Bmatrix} A & B & C \\ a & b & c \end{Bmatrix} \quad (\text{B.22})$$

recast (B.20) as

$$\hat{C}_{J_2 m_2(jj) J_3 m_3(jj)}^{J_1 m_1(jj)} = (-1)^{J_1+2j} \sqrt{n(2J_2+1)(2J_3+1)} C_{J_2 m_2 J_3 m_3}^{J_1 m_1} \begin{Bmatrix} J_1 & J_2 & J_3 \\ j & j & j \end{Bmatrix}. \quad (\text{B.23})$$

The first equality of (B.13) leads to

$$\hat{C}_{J_2 m_2(jj) J_3 m_3(jj)}^{J_1 m_1(jj)} = (-1)^{J_2+J_3-J_1} \hat{C}_{J_3 m_3(jj) J_2 m_2(jj)}^{J_1 m_1(jj)}. \quad (\text{B.24})$$

B.2 Spin- S fuzzy spherical harmonics

Spin- S fuzzy spherical harmonics is defined by

$$\hat{\mathcal{Y}}_{J m, \tilde{J}(jj)}^{S n'} \equiv \sum_{p=-\tilde{J}}^{\tilde{J}} C_{\tilde{J} p S n'}^{J m} \hat{Y}_{\tilde{J} p}^{(jj)} = C_{\tilde{J} m-n' S n'}^{J m} \hat{Y}_{\tilde{J} m-n'}^{(jj)}, \quad (\text{B.25})$$

where $\hat{Y}_{\tilde{J} m-n'}^{(jj)}$ stands for a harmonics of the orbital angular momentum $(\tilde{J}, m-n')$ on the fuzzy S^2 . Combined with a wave function with spin S , $\chi_{S n'}$, which satisfies

$$\vec{S}^2 \chi_{S n'} = S(S+1) \chi_{S n'}, \quad S_z \chi_{S n'} = n' \chi_{S n'}, \quad (\text{B.26})$$

$\hat{\mathcal{Y}}_{J m, \tilde{J}(jj)}^{S n'} \chi_{S n'}$ represents the irreducible representation of the total angular momentum (J, m) obtained from the tensor product $(\tilde{J}, m-n') \otimes (S, n')$.

In the text, S_{\pm}, S_3 are related to the $SU(2)_R$ generators (2.11) as

$$S_{\pm} = J_{\pm\pm}, \quad S_z = \frac{1}{2} J_0. \quad (\text{B.27})$$

$\chi_{S n'}$ comes from the wave functions of the doublets $(\psi_{+\mu}, \psi_{-\mu})$, $(\rho_{+\underline{i}}, \rho_{-\underline{i}})$, $(\chi_{+\mathbf{A}}, \chi_{-\mathbf{A}})$, $(\eta_+, -\eta_-)$ for the $S = \frac{1}{2}$ case, and from the wave functions of the triplet $(\frac{1}{\sqrt{2}}\phi_+, \frac{1}{2}C, \frac{1}{\sqrt{2}}\phi_-)$ for the $S = 1$ case.

Scalar fuzzy spherical harmonics For $S = 0$, (B.25) reduces to the scalar fuzzy spherical harmonics previously discussed:

$$\hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{00} = C_{Jm00}^{Jm} \hat{Y}_{Jm}^{(jj)} = \delta_{J\tilde{J}} \hat{Y}_{Jm}^{(jj)}. \quad (\text{B.28})$$

Vector fuzzy spherical harmonics For $S = 1$, $\hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{1n'}$ ($n' = 1, 0, -1$) are used for the mode expansion of $\frac{1}{\sqrt{2}}\phi_+$, $\frac{1}{2}C$, $-\frac{1}{\sqrt{2}}\phi_-$, respectively.

On the other hand, from (2.7) the following basis $\vec{Y}_{Jm(jj)}^\rho$ ($\rho = 1, 0, -1$) is convenient to expand $\vec{Y} \equiv (X_9, X_{10}, X_8)^T$ in modes:

$$\vec{Y}_{Jm(jj)}^{\rho=1} = i\vec{\mathcal{Y}}_{J+1m, J(jj)}, \quad \vec{Y}_{Jm(jj)}^{\rho=-1} = -i\vec{\mathcal{Y}}_{Jm, J+1(jj)}, \quad \vec{Y}_{Jm(jj)}^{\rho=0} = \vec{\mathcal{Y}}_{Jm, J(jj)} \quad (\text{B.29})$$

with

$$\vec{\mathcal{Y}}_{Jm, \tilde{J}(jj)} = \begin{pmatrix} \hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{i=1} \\ \hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{i=2} \\ \hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{i=3} \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -\hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{11} + \hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{1-1} \\ -i\hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{11} - i\hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{1-1} \\ \sqrt{2}\hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{10} \end{pmatrix}. \quad (\text{B.30})$$

More explicitly,

$$\vec{Y}_{Jm(jj)}^{\rho=1} = iV \begin{pmatrix} \hat{\mathcal{Y}}_{J+1m, J(jj)}^{11} \\ \hat{\mathcal{Y}}_{J+1m, J(jj)}^{10} \\ \hat{\mathcal{Y}}_{J+1m, J(jj)}^{1-1} \end{pmatrix} = iV \begin{pmatrix} C_{Jm-111}^{J+1m} \hat{Y}_{Jm-1}^{(jj)} \\ C_{Jm10}^{J+1m} \hat{Y}_{Jm}^{(jj)} \\ C_{Jm+11-1}^{J+1m} \hat{Y}_{Jm+1}^{(jj)} \end{pmatrix}, \quad (\text{B.31})$$

$$\vec{Y}_{Jm(jj)}^{\rho=0} = V \begin{pmatrix} \hat{\mathcal{Y}}_{Jm, J(jj)}^{11} \\ \hat{\mathcal{Y}}_{Jm, J(jj)}^{10} \\ \hat{\mathcal{Y}}_{Jm, J(jj)}^{1-1} \end{pmatrix} = V \begin{pmatrix} C_{Jm-111}^{Jm} \hat{Y}_{Jm-1}^{(jj)} \\ C_{Jm10}^{Jm} \hat{Y}_{Jm}^{(jj)} \\ C_{Jm+11-1}^{Jm} \hat{Y}_{Jm+1}^{(jj)} \end{pmatrix}, \quad (\text{B.32})$$

$$\vec{Y}_{Jm(jj)}^{\rho=-1} = -iV \begin{pmatrix} \hat{\mathcal{Y}}_{Jm, J+1(jj)}^{11} \\ \hat{\mathcal{Y}}_{Jm, J+1(jj)}^{10} \\ \hat{\mathcal{Y}}_{Jm, J+1(jj)}^{1-1} \end{pmatrix} = -iV \begin{pmatrix} C_{J+1m-111}^{Jm} \hat{Y}_{J+1m-1}^{(jj)} \\ C_{J+1m10}^{Jm} \hat{Y}_{J+1m}^{(jj)} \\ C_{J+1m+11-1}^{Jm} \hat{Y}_{J+1m+1}^{(jj)} \end{pmatrix}, \quad (\text{B.33})$$

where V is a unitary matrix

$$V \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \quad (\text{B.34})$$

with $\det V = -i$.

We can see that $J, m \in \mathbb{Z}$ in $\vec{Y}_{Jm(jj)}^\rho$ from (B.31)–(B.33). Also, for the $J = 0$ case, $\hat{\mathcal{Y}}_{0m, 0(jj)}^{1n'} = 0$ because of $C_{0m1n'}^{0m} = 0$. Thus,

$$\vec{Y}_{0m(jj)}^{\rho=0} = 0. \quad (\text{B.35})$$

Spinor fuzzy spherical harmonics For $S = \frac{1}{2}$, the spinor fuzzy spherical harmonics

$$\hat{Y}_{Jm(jj)\alpha}^{\kappa} \quad \left(\kappa = 1, -1, \quad \alpha = \frac{1}{2}, -\frac{1}{2} \right) \quad (\text{B.36})$$

is defined as

$$\hat{Y}_{Jm(jj)\alpha}^{\kappa=1} \equiv \hat{\mathcal{Y}}_{J+\frac{1}{2}m, J(jj)}^{S=\frac{1}{2}\alpha} = C_{Jm-\alpha\frac{1}{2}\alpha}^{J+\frac{1}{2}m} \hat{Y}_{Jm-\alpha}^{(jj)}, \quad (\text{B.37})$$

$$\hat{Y}_{Jm(jj)\alpha}^{\kappa=-1} \equiv \hat{\mathcal{Y}}_{Jm, J+\frac{1}{2}(jj)}^{S=\frac{1}{2}\alpha} = C_{J+\frac{1}{2}m-\alpha\frac{1}{2}\alpha}^{Jm} \hat{Y}_{J+\frac{1}{2}m-\alpha}^{(jj)}. \quad (\text{B.38})$$

Here, κ labels the spinor basis, and α labels the spinor components for each basis.

Note that $m \in \mathbb{Z} + \frac{1}{2}$ for $\hat{Y}_{Jm(jj)\alpha}^{\kappa=1}$, because J runs integers in $\hat{Y}_{Jm-\alpha}^{(jj)}$. On the other hand, $J \in \mathbb{Z} + \frac{1}{2}$ and $m \in \mathbb{Z} + \frac{1}{2}$ for $\hat{Y}_{Jm(jj)\alpha}^{\kappa=-1}$.

B.3 Hermitian conjugates

From (B.25), the hermitian conjugate of the spin- S fuzzy spherical harmonics $\hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{S n'}$ reads

$$\begin{aligned} \left(\hat{\mathcal{Y}}_{Jm, \tilde{J}(jj)}^{S n'} \right)^{\dagger} &= (-1)^{-J+\tilde{J}+S+m-n'} C_{\tilde{J}-m+n' S-n'}^{J-m} \hat{Y}_{\tilde{J}-m+n'}^{(jj)} \\ &= (-1)^{-J+\tilde{J}-S+m+n'} \hat{\mathcal{Y}}_{J-m, \tilde{J}(jj)}^{S-n'}, \end{aligned} \quad (\text{B.39})$$

where we have used (B.13), (B.14) and $S-n' \in \mathbb{Z}$.

The vector fuzzy spherical harmonics with $\rho = 1$ (B.31),

$$\hat{Y}_{Jm(jj)i}^{\rho=1} = i \sum_{n'=-1}^1 V_{in'} \hat{\mathcal{Y}}_{J+1m, J(jj)}^{1 n'}, \quad (\text{B.40})$$

has the hermitian conjugate as

$$\left(\hat{Y}_{Jm(jj)i}^{\rho=1} \right)^{\dagger} = (-1)^{m+1} \hat{Y}_{J-m(jj)i}^{\rho=1}, \quad (\text{B.41})$$

which can be seen from (B.39) and the identity

$$V_{in'}^* (-1)^{-n'} = V_{i-n'}. \quad (\text{B.42})$$

Repeating a similar argument for (B.32) and (B.33), we conclude that

$$\left(\hat{Y}_{Jm(jj)i}^{\rho} \right)^{\dagger} = (-1)^{m+1} \hat{Y}_{J-m(jj)i}^{\rho}. \quad (\text{B.43})$$

For the spinor fuzzy spherical harmonics (B.37) and (B.38), their hermitian conjugates turn out to be

$$\left(\hat{Y}_{Jm(jj)\alpha}^{\kappa} \right)^{\dagger} = (-1)^{m+1+\kappa\alpha} \hat{Y}_{J-m(jj)-\alpha}^{\kappa}. \quad (\text{B.44})$$

B.4 Orthonormality

From (B.25) and (B.15), we have

$$\frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{J' m_1, \tilde{J}_1(jj)}^{S n'} \right)^\dagger \hat{\mathcal{Y}}_{J'' m_2, \tilde{J}_2(jj)}^{S n'} \right\} = \delta_{\tilde{J}_1 \tilde{J}_2} \delta_{m_1 m_2} C_{\tilde{J}_1 m_1 - n' S n'}^{J' m_1} C_{\tilde{J}_1 m_1 - n' S n'}^{J'' m_1}. \quad (\text{B.45})$$

Taking the sum over n' leads to

$$\sum_{n'=-S}^S \frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{J' m_1, \tilde{J}_1(jj)}^{S n'} \right)^\dagger \hat{\mathcal{Y}}_{J'' m_2, \tilde{J}_2(jj)}^{S n'} \right\} = \delta_{J' J''} \delta_{\tilde{J}_1 \tilde{J}_2} \delta_{m_1 m_2}, \quad (\text{B.46})$$

where we have used ¹³

$$\sum_{n'=-S}^S C_{\tilde{J}_1 m_1 - n' S n'}^{J' m_1} C_{\tilde{J}_1 m_1 - n' S n'}^{J'' m_1} = \sum_{m'=-\tilde{J}_1}^{\tilde{J}_1} \sum_{n'=-S}^S C_{\tilde{J}_1 m' S n'}^{J' m_1} C_{\tilde{J}_1 m' S n'}^{J'' m_1} = \delta_{J' J''}. \quad (\text{B.47})$$

For the vector fuzzy spherical harmonics (B.31), the identity

$$\sum_{i=1}^3 V_{i n'_1}^* V_{i n'_2} = \delta_{n'_1 n'_2} \quad (V \text{ is unitary}) \quad (\text{B.48})$$

and (B.46) imply

$$\sum_{i=1}^3 \frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{J' m_1(jj)i}^{\rho=1} \right)^\dagger \hat{Y}_{J'' m_2(jj)i}^{\rho=1} \right\} = \delta_{J' J''} \delta_{m_1 m_2}. \quad (\text{B.49})$$

For cases including (B.32) and (B.33), similar formulas are obtained, and we conclude that

$$\sum_{i=1}^3 \frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{J' m_1(jj)i}^{\rho_1} \right)^\dagger \hat{Y}_{J'' m_2(jj)i}^{\rho_2} \right\} = \delta_{\rho_1 \rho_2} \delta_{J' J''} \delta_{m_1 m_2}. \quad (\text{B.50})$$

For the spinor fuzzy spherical harmonics (B.37) and (B.38),

$$\sum_{\alpha=-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{J' m_1(jj)\alpha}^{\kappa_1} \right)^\dagger \hat{Y}_{J'' m_2(jj)\alpha}^{\kappa_2} \right\} = \delta_{\kappa_1 \kappa_2} \delta_{J' J''} \delta_{m_1 m_2} \quad (\text{B.51})$$

holds.

¹³Note that the sum over m' in the middle of (B.47) is trivial because of the momentum conservation $m_1 = m' + n'$. The second equality is nothing but (B.17).

B.5 Some C-G coefficients

The C-G coefficient is related to the $3j$ symbol as

$$C_{J' m_1 J'' m_2}^{J m} = (-1)^{-J'+J''-m} \sqrt{2J+1} \begin{pmatrix} J' & J'' & J \\ m_1 & m_2 & -m \end{pmatrix}. \quad (\text{B.52})$$

Here we present the explicit form of some C-G coefficients that will be used later.

$$C_{J m-1 1 1}^{J+1 m} = \sqrt{\frac{(J+m+1)(J+m)}{2(2J+1)(J+1)}}, \quad (\text{B.53})$$

$$C_{J m 1 0}^{J+1 m} = \sqrt{\frac{(J+m+1)(J-m+1)}{(2J+1)(J+1)}}, \quad (\text{B.54})$$

$$C_{J m+1 1 -1}^{J+1 m} = \sqrt{\frac{(J-m+1)(J-m)}{2(2J+1)(J+1)}}, \quad (\text{B.55})$$

$$C_{J+1 m-1 1 1}^J = \sqrt{\frac{(J-m+2)(J-m+1)}{2(2J+3)(J+1)}}, \quad (\text{B.56})$$

$$C_{J+1 m 1 0}^J = -\sqrt{\frac{(J-m+1)(J+m+1)}{(2J+3)(J+1)}}, \quad (\text{B.57})$$

$$C_{J+1 m+1 1 -1}^J = \sqrt{\frac{(J+m+2)(J+m+1)}{2(2J+3)(J+1)}}, \quad (\text{B.58})$$

$$C_{J m-1 1 1}^J = -\sqrt{\frac{(J+m)(J-m+1)}{2(J+1)J}}, \quad (\text{B.59})$$

$$C_{J m 1 0}^J = \frac{m}{\sqrt{(J+1)J}}, \quad (\text{B.60})$$

$$C_{J m+1 1 -1}^J = \sqrt{\frac{(J-m)(J+m+1)}{2(J+1)J}}, \quad (\text{B.61})$$

$$C_{J m-\frac{1}{2} \frac{1}{2} \frac{1}{2}}^{J+\frac{1}{2} m} = \sqrt{\frac{J+m+\frac{1}{2}}{2J+1}}, \quad C_{J m+\frac{1}{2} \frac{1}{2} -\frac{1}{2}}^{J+\frac{1}{2} m} = \sqrt{\frac{J-m+\frac{1}{2}}{2J+1}}, \quad (\text{B.62})$$

$$C_{J+\frac{1}{2} m-\frac{1}{2} \frac{1}{2} \frac{1}{2}}^J = -\sqrt{\frac{J-m+1}{2(J+1)}}, \quad C_{J+\frac{1}{2} m+\frac{1}{2} \frac{1}{2} -\frac{1}{2}}^J = \sqrt{\frac{J+m+1}{2(J+1)}}. \quad (\text{B.63})$$

B.6 \vec{L} -actions to scalar, vector and spinor fuzzy spherical harmonics

In this subsection, we will show that the following four relations hold:

$$\vec{L} \circ \hat{Y}_{Jm}^{(jj)} = \sqrt{J(J+1)} \vec{\hat{Y}}_{Jm(jj)}^{\rho=0}, \quad (\text{B.64})$$

$$\vec{L} \circ \cdot \vec{\hat{Y}}_{Jm(jj)}^\rho = \sqrt{J(J+1)} \delta_{\rho 0} \hat{Y}_{Jm}^{(jj)}, \quad (\text{B.65})$$

$$i\vec{L} \circ \times \vec{\hat{Y}}_{Jm(jj)}^\rho + \vec{\hat{Y}}_{Jm(jj)}^\rho = \rho(J+1) \vec{\hat{Y}}_{Jm(jj)}^\rho, \quad (\text{B.66})$$

$$\left(\vec{\sigma} \cdot \vec{L} \circ + \frac{3}{4} \right) \hat{Y}_{Jm(jj)}^\kappa = \kappa \left(J + \frac{3}{4} \right) \hat{Y}_{Jm(jj)}^\kappa, \quad (\text{B.67})$$

where

$$\hat{Y}_{Jm(jj)}^\kappa = \begin{pmatrix} \hat{Y}_{Jm(jj)\alpha=\frac{1}{2}}^\kappa \\ \hat{Y}_{Jm(jj)\alpha=-\frac{1}{2}}^\kappa \end{pmatrix}, \quad (\text{B.68})$$

$\vec{\sigma}$ are the Pauli matrices, and

$$\vec{\sigma} \cdot \vec{L} = \begin{pmatrix} L_3 & L_1 - iL_2 \\ L_1 + iL_2 & -L_3 \end{pmatrix} = \begin{pmatrix} L_3 & L_- \\ L_+ & -L_3 \end{pmatrix}. \quad (\text{B.69})$$

Proof of (B.64) Note that

$$\begin{aligned} & \sqrt{J(J+1)} \left(\hat{Y}_{Jm(jj)i=1}^{\rho=0} + i\hat{Y}_{Jm(jj)i=2}^{\rho=0} \right) \\ &= \sqrt{2J(J+1)} \hat{\mathcal{Y}}_{Jm,J}^{1-1} = \sqrt{2J(J+1)} C_{Jm+1,1-1}^{Jm} \hat{Y}_{Jm+1}^{(jj)}. \end{aligned} \quad (\text{B.70})$$

Using (B.61), we can see

$$\begin{aligned} \sqrt{J(J+1)} \left(\hat{Y}_{Jm(jj)i=1}^{\rho=0} + i\hat{Y}_{Jm(jj)i=2}^{\rho=0} \right) &= \sqrt{(J-m)(J+m+1)} \hat{Y}_{Jm+1}^{(jj)} \\ &= L_+ \circ \hat{Y}_{Jm}^{(jj)}. \end{aligned} \quad (\text{B.71})$$

Similarly, we obtain

$$\sqrt{J(J+1)} \left(\hat{Y}_{Jm(jj)i=1}^{\rho=0} - i\hat{Y}_{Jm(jj)i=2}^{\rho=0} \right) = L_- \circ \hat{Y}_{Jm}^{(jj)}, \quad (\text{B.72})$$

$$\sqrt{J(J+1)} \hat{Y}_{Jm(jj)i=3}^{\rho=0} = L_3 \circ \hat{Y}_{Jm}^{(jj)}. \quad (\text{B.73})$$

(B.71), (B.72) and (B.73) mean (B.64).

Proof of (B.65) For $\rho = 0$, acting $\vec{L} \circ \cdot$ on (B.64) leads to

$$J(J+1) \hat{Y}_{Jm}^{(jj)} = \sqrt{J(J+1)} \vec{L} \circ \cdot \hat{Y}_{Jm(jj)}^{\rho=0}, \quad (\text{B.74})$$

which is nothing but (B.65) when $J \neq 0$. At $J = 0$, (B.65) trivially holds because of (B.35).

For $\rho = 1$,

$$\begin{aligned} \vec{L} \circ \cdot \hat{Y}_{Jm(jj)}^{\rho=1} &= \frac{1}{2} L_+ \circ \left(\hat{Y}_{Jm(jj)i=1}^{\rho=1} - i \hat{Y}_{Jm(jj)i=2}^{\rho=1} \right) + \frac{1}{2} L_- \circ \left(\hat{Y}_{Jm(jj)i=1}^{\rho=1} + i \hat{Y}_{Jm(jj)i=2}^{\rho=1} \right) \\ &\quad + L_3 \circ \hat{Y}_{Jm(jj)i=3}^{\rho=1} \\ &= -\frac{i}{\sqrt{2}} L_+ \circ \hat{\mathcal{Y}}_{J+1m, J(jj)}^{11} + \frac{i}{\sqrt{2}} L_- \circ \hat{\mathcal{Y}}_{J+1m, J(jj)}^{1-1} + i L_3 \circ \hat{\mathcal{Y}}_{J+1m, J(jj)}^{10} \\ &= -\frac{i}{\sqrt{2}} C_{Jm-111}^{J+1m} L_+ \circ \hat{Y}_{Jm-1}^{(jj)} + \frac{i}{\sqrt{2}} C_{Jm+11-1}^{J+1m} L_- \circ \hat{Y}_{Jm+1}^{(jj)} + i C_{Jm10}^{J+1m} L_3 \circ \hat{Y}_{Jm}^{(jj)} \\ &= \left[-\frac{i}{\sqrt{2}} C_{Jm-111}^{J+1m} \sqrt{(J-m+1)(J+m)} \right. \\ &\quad \left. + \frac{i}{\sqrt{2}} C_{Jm+11-1}^{J+1m} \sqrt{(J+m+1)(J-m)} + i C_{Jm10}^{J+1m} m \right] \hat{Y}_{Jm}^{(jj)}. \end{aligned} \quad (\text{B.75})$$

By using (B.53)–(B.55), it turns out that the r.h.s. of (B.75) vanishes.

Finally, $\vec{L} \circ \cdot \hat{Y}_{Jm(jj)}^{\rho=-1} = 0$ is similarly shown, which completes the proof of (B.65).

Proof of (B.66) When $J = 0$, (B.66) trivially holds from (B.35). Let us focus on the case $J \neq 0$.

For $\rho = 0$, acting $\vec{L} \circ \times$ on (B.64) yields

$$i \vec{L} \circ \times \hat{Y}_{Jm(jj)}^{\rho=0} = \frac{i}{\sqrt{J(J+1)}} \vec{L} \circ \times \vec{L} \circ \hat{Y}_{Jm}^{(jj)} = -\frac{1}{\sqrt{J(J+1)}} \vec{L} \circ \hat{Y}_{Jm}^{(jj)} = -\hat{Y}_{Jm(jj)}^{\rho=0}. \quad (\text{B.76})$$

Since

$$\vec{L} \circ \times \vec{L} \circ = \begin{pmatrix} L_2 \circ L_3 \circ - L_3 \circ L_2 \circ \\ L_3 \circ L_1 \circ - L_1 \circ L_3 \circ \\ L_1 \circ L_2 \circ - L_2 \circ L_1 \circ \end{pmatrix} = i \begin{pmatrix} L_1 \circ \\ L_2 \circ \\ L_3 \circ \end{pmatrix} = i \vec{L} \circ, \quad (\text{B.77})$$

(B.76) proves (B.66) for $\rho = 0$.

For $\rho = 1$, let us consider, e.g., the $i = 3$ component:

$$\begin{aligned} &i \left(L_1 \circ \hat{Y}_{Jm(jj)i=2}^{\rho=1} - L_2 \circ \hat{Y}_{Jm(jj)i=1}^{\rho=1} \right) \\ &= -\frac{1}{\sqrt{2}} \left[L_1 \circ \left(-i \hat{\mathcal{Y}}_{J+1m, J(jj)}^{11} - i \hat{\mathcal{Y}}_{J+1m, J(jj)}^{1-1} \right) - L_2 \circ \left(-\hat{\mathcal{Y}}_{J+1m, J(jj)}^{11} + \hat{\mathcal{Y}}_{J+1m, J(jj)}^{1-1} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{i}{\sqrt{2}} \left(L_+ \circ \hat{\mathcal{Y}}_{J+1m, J(jj)}^{11} + L_- \circ \hat{\mathcal{Y}}_{J+1m, J(jj)}^{1-1} \right) \\
&= \frac{i}{\sqrt{2}} \left(C_{Jm-111}^{J+1m} L_+ \circ \hat{Y}_{Jm-1}^{(jj)} + C_{Jm+11-1}^{J+1m} L_- \circ \hat{Y}_{Jm+1}^{(jj)} \right). \tag{B.78}
\end{aligned}$$

With use of (B.53)–(B.55), the r.h.s. of (B.78) becomes

$$(\text{r.h.s. of (B.78)}) = iJC_{Jm10}^{J+1m} \hat{Y}_{Jm}^{(jj)} = iJ\hat{\mathcal{Y}}_{J+1m, J(jj)}^{10} = J\hat{Y}_{Jm(jj)i=3}^{\rho=1}, \tag{B.79}$$

which shows (B.66) for $\rho = 1$ and $i = 3$.

We can similarly show (B.66) for the remaining cases including $\rho = -1$.

Proof of (B.67) For $\kappa = 1$, let us consider, e.g., the $\alpha = \frac{1}{2}$ component:

$$\begin{aligned}
\left(\left(\vec{\sigma} \cdot \vec{L} \circ + \frac{3}{4} \right) \hat{Y}_{Jm(jj)}^{\kappa=1} \right)_{\alpha=\frac{1}{2}} &= \left(L_3 \circ + \frac{3}{4} \right) \hat{Y}_{Jm(jj)\alpha=\frac{1}{2}}^{\kappa=1} + L_- \circ \hat{Y}_{Jm(jj)\alpha=-\frac{1}{2}}^{\kappa=1} \\
&= C_{Jm-\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{J+\frac{1}{2}m} \left(L_3 \circ + \frac{3}{4} \right) \hat{Y}_{Jm-\frac{1}{2}}^{(jj)} + C_{Jm+\frac{1}{2}\frac{1}{2}-\frac{1}{2}}^{J+\frac{1}{2}m} L_- \circ \hat{Y}_{Jm+\frac{1}{2}}^{(jj)}. \tag{B.80}
\end{aligned}$$

By using (B.62), the r.h.s. of (B.80) can be expressed as

$$(\text{r.h.s. of (B.80)}) = \left(J + \frac{3}{4} \right) C_{Jm-\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{J+\frac{1}{2}m} \hat{Y}_{Jm-\frac{1}{2}}^{(jj)} = \left(J + \frac{3}{4} \right) \hat{Y}_{Jm(jj)\alpha=\frac{1}{2}}^{\kappa=1}, \tag{B.81}$$

showing (B.67) for $\kappa = 1$ and $\alpha = \frac{1}{2}$.

We can similarly prove (B.67) for all the other cases.

B.7 Vertex coefficients

We start by showing the formula for the trace of the product of three kinds of spin- S fuzzy spherical harmonics:

$$\begin{aligned}
&\sum_{n_1, n_2, n_3} \frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{J_1 m_1, \tilde{J}_1(jj)}^{S_1 n_1} \right)^\dagger \hat{\mathcal{Y}}_{J_2 m_2, \tilde{J}_2(jj)}^{S_2 n_2} \hat{\mathcal{Y}}_{J_3 m_3, \tilde{J}_3(jj)}^{S_3 n_3} \right\} C_{S_2 n_2 S_3 n_3}^{S_1 n_1} \\
&= (-1)^{\tilde{J}_1 + 2j} \sqrt{n(2S_1 + 1)(2\tilde{J}_1 + 1)(2J'' + 1)(2\tilde{J}_2 + 1)(2J_3 + 1)(2\tilde{J}_3 + 1)} \\
&\quad \times \begin{Bmatrix} J_1 & \tilde{J}_1 & S_1 \\ J_2 & \tilde{J}_2 & S_2 \\ J_3 & \tilde{J}_3 & S_3 \end{Bmatrix} C_{J_2 m_2 J_3 m_3}^{J_1 m_1} \begin{Bmatrix} \tilde{J}_1 & \tilde{J}_2 & \tilde{J}_3 \\ j & j & j \end{Bmatrix}. \tag{B.82}
\end{aligned}$$

From the definition (B.25) and (B.23),

(l.h.s. of (B.82))

$$\begin{aligned}
&= \sum_{n_1, n_2, n_3} C_{\tilde{J}_1 m_1 - n_1 S_1 n_1}^{J_1 m_1} C_{\tilde{J}_2 m_2 - n_2 S_2 n_2}^{J_2 m_2} C_{\tilde{J}_3 m_3 - n_3 S_3 n_3}^{J_3 m_3} C_{S_2 n_2 S_3 n_3}^{S_1 n_1} \hat{C}_{\tilde{J}_2 m_2 - n_2(jj) \tilde{J}_3 m_3 - n_3(jj)}^{\tilde{J}_1 m_1 - n_1(jj)} \\
&= \sum_{p_1, p_2, p_3} \sum_{n_1, n_2, n_3} C_{\tilde{J}_1 p_1 S_1 n_1}^{J_1 m_1} C_{\tilde{J}_2 p_2 S_2 n_2}^{J_2 m_2} C_{\tilde{J}_3 p_3 S_3 n_3}^{J_3 m_3} C_{S_2 n_2 S_3 n_3}^{S_1 n_1} \hat{C}_{\tilde{J}_2 p_2(jj) \tilde{J}_3 p_3(jj)}^{\tilde{J}_1 p_1(jj)} \\
&= \sum_{p_3, n_3} \left[\sum_{p_1, n_1, p_2, n_2} C_{\tilde{J}_1 p_1 S_1 n_1}^{J_1 m_1} C_{\tilde{J}_2 p_2 S_2 n_2}^{J_2 m_2} C_{\tilde{J}_2 p_2 \tilde{J}_3 p_3}^{\tilde{J}_1 p_1} C_{S_2 n_2 S_3 n_3}^{S_1 n_1} \right] \\
&\quad \times C_{\tilde{J}_3 p_3 S_3 n_3}^{J_3 m_3} (-1)^{\tilde{J}_1 + 2j} \sqrt{n(2\tilde{J}_2 + 1)(2\tilde{J}_3 + 1)} \begin{Bmatrix} \tilde{J}_1 & \tilde{J}_2 & \tilde{J}_3 \\ j & j & j \end{Bmatrix}. \tag{B.83}
\end{aligned}$$

Applying the identity ¹⁴

$$\begin{aligned}
&\sum_{\beta, \gamma, \epsilon, \varphi} C_{b\beta c\gamma}^{a\alpha} C_{e\epsilon f\varphi}^{d\delta} C_{e\epsilon g\eta}^{b\beta} C_{f\varphi j\mu}^{c\gamma} \\
&= \sum_{k, \kappa} \sqrt{(2b+1)(2c+1)(2d+1)(2k+1)} C_{g\eta j\mu}^{k\kappa} C_{d\delta k\kappa}^{a\alpha} \begin{Bmatrix} a & b & c \\ d & e & f \\ k & g & j \end{Bmatrix} \tag{B.84}
\end{aligned}$$

to $[\sum_{p_1, n_1, p_2, n_2} \dots]$ in (B.83) together with the orthogonality (B.17), we obtain (B.82).

To derive formulas for the traces including the vector and spinor fuzzy spherical harmonics, it is convenient to recast the expressions (B.31)–(B.33) as well as (B.37) and (B.38) into the concise form:

$$\hat{Y}_{Jm(jj)i}^\rho = i^\rho \sum_{n'=-1}^1 V_{in'} \hat{\mathcal{Y}}_{Qm, \tilde{Q}(jj)}^{1n'} \tag{B.85}$$

with $Q \equiv J + \delta_{\rho, 1}$, $\tilde{Q} \equiv J + \delta_{\rho, -1}$, and

$$\hat{Y}_{Jm(jj)\alpha}^\kappa = \hat{\mathcal{Y}}_{Um, \tilde{U}(jj)}^{\frac{1}{2}\alpha} \tag{B.86}$$

with $U \equiv J + \frac{1}{2}\delta_{\kappa, 1}$, $\tilde{U} \equiv J + \frac{1}{2}\delta_{\kappa, -1}$. In the remaining part of this section, we compute the vertex coefficients defined by (3.19)–(3.22) in the text.

B.7.1 $\hat{\mathcal{D}}$

We plug (B.85) into (3.19) and note that

$$\sum_{i=1}^3 V_{in_1} V_{i-n_2} = \sum_{i=1}^3 V_{in_1} V_{in_2}^* (-1)^{-n_2} = \delta_{n_1 n_2} (-1)^{-n_2}, \tag{B.87}$$

¹⁴See, e.g., eq. (26) in Chapter 8.7.4 of [58].

which follows from (B.42), so that the $\hat{\mathcal{D}}$ coefficient is expressed as

$$\begin{aligned} & \hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{J m(jj)} \\ &= i^{\rho_1 + \rho_2} \sum_{n_1=-1}^1 (-1)^{-n_1} \frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{J m, J(jj)}^{00} \right)^\dagger \hat{\mathcal{Y}}_{Q_1 m_1, \tilde{Q}_1(jj)}^{1 n_1} \hat{\mathcal{Y}}_{Q_2 m_2, \tilde{Q}_2(jj)}^{1 - n_1} \right\} \end{aligned} \quad (\text{B.88})$$

with $Q_a = J_a + \delta_{\rho_a, 1}$, $\tilde{Q}_a = J_a + \delta_{\rho_a, -1}$ ($a = 1, 2$).

Next, we use (B.39) to rewrite it as

$$\begin{aligned} & \hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{J m(jj)} \\ &= i^{\rho_1 + \rho_2} (-1)^{Q_1 - \tilde{Q}_1 + 1 + m_1 + m} \sum_{n_1=-1}^1 \frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{Q_1 - m_1, \tilde{Q}_1(jj)}^{1 n_1} \right)^\dagger \hat{\mathcal{Y}}_{Q_2 m_2, \tilde{Q}_2(jj)}^{1 n_1} \hat{\mathcal{Y}}_{J - m, J(jj)}^{00} \right\} \\ &= i^{\rho_1 + \rho_2} (-1)^{Q_1 - \tilde{Q}_1 + 1 + m_1 + m} \\ &\quad \times \sum_{n_1, n_2=-1}^1 \frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{Q_1 - m_1, \tilde{Q}_1(jj)}^{1 n_1} \right)^\dagger \hat{\mathcal{Y}}_{Q_2 m_2, \tilde{Q}_2(jj)}^{1 n_2} \hat{\mathcal{Y}}_{J - m, J(jj)}^{00} \right\} C_{1 n_2 0 0}^{1 n_1}. \end{aligned} \quad (\text{B.89})$$

In the last equality, we have inserted $1 = \sum_{n_2} \delta_{n_1 n_2} = \sum_{n_2} C_{1 n_2 0 0}^{1 n_1}$.

Then, applying the formula (B.82) and

$$\begin{aligned} C_{Q_2 m_2 J - m}^{Q_1 - m_1} &= (-1)^{Q_2 - m_2} \sqrt{\frac{2Q_1 + 1}{2J + 1}} C_{Q_1 - m_1 Q_2 - m_2}^{J - m} \\ &= (-1)^{Q_2 - m_2} \sqrt{\frac{2Q_1 + 1}{2J + 1}} \cdot (-1)^{Q_1 + Q_2 - J} C_{Q_1 m_1 Q_2 m_2}^{J m} \end{aligned} \quad (\text{B.90})$$

yields

$$\begin{aligned} \hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{J m(jj)} &= i^{\rho_1 + \rho_2} (-1)^{2Q_1 + 2Q_2 + 2j - J + 1 + m_1 - m_2 + m} \\ &\quad \times \sqrt{3n(2Q_1 + 1)(2\tilde{Q}_1 + 1)(2Q_2 + 1)(2\tilde{Q}_2 + 1)(2J + 1)} \\ &\quad \times \begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & 1 \\ J & J & 0 \end{Bmatrix} C_{Q_1 m_1 Q_2 m_2}^{J m} \begin{Bmatrix} \tilde{Q}_1 & \tilde{Q}_2 & J \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{B.91})$$

We here note that J, J_1, J_2, m, m_1, m_2 are integers, $m = m_1 + m_2$ from the C-G coefficient $C_{Q_1 m_1 Q_2 m_2}^{J m}$,

$$(2Q_a + 1)(2\tilde{Q}_a + 1) = (2J_a + 1)(2J_a + 2\rho_a^2 + 1) \quad (a = 1, 2), \quad (\text{B.92})$$

the symmetric property of the $6j$ symbol

$$\begin{Bmatrix} a & b & c \\ A & B & C \end{Bmatrix} = \begin{Bmatrix} b & a & c \\ B & A & C \end{Bmatrix} = \begin{Bmatrix} a & c & b \\ A & C & B \end{Bmatrix}, \quad (\text{B.93})$$

and the relation ¹⁵

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & 0 \end{Bmatrix} = \delta_{cf} \delta_{gh} \frac{(-1)^{b+c+d+g}}{\sqrt{(2c+1)(2g+1)}} \begin{Bmatrix} a & b & c \\ e & d & g \end{Bmatrix}. \quad (\text{B.94})$$

These reduce (B.91) to

$$\begin{aligned} & \hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{J m(jj)} \\ &= i^{\rho_1+\rho_2} (-1)^{2j+\tilde{Q}_1+Q_2} \sqrt{n(2J_1+1)(2J_1+2\rho_1^2+1)(2J_2+1)(2J_2+2\rho_2^2+1)} \\ & \times C_{Q_1 m_1 Q_2 m_2}^{J m} \begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ \tilde{Q}_2 & Q_2 & J \end{Bmatrix} \begin{Bmatrix} J & \tilde{Q}_1 & \tilde{Q}_2 \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{B.95})$$

Equivalently, we can rewrite (3.19) as

$$\sum_{i=1}^3 \hat{Y}_{J_1 m_1(jj) i}^{\rho_1} \hat{Y}_{J_2 m_2(jj) i}^{\rho_2} = \sum_{J=0}^{2j} \sum_{m=-J}^J \hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{J m(jj)} \hat{Y}_{J m}^{(jj)}, \quad (\text{B.96})$$

and

$$\left(\hat{Y}_{J m}^{(jj)} \right)^\dagger \hat{Y}_{J_1 m_1(jj) i}^{\rho_1} = \sum_{\rho_2, J_2, m_2} \hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{J m(jj)} \left(\hat{Y}_{J_2 m_2(jj) i}^{\rho_2} \right)^\dagger, \quad (\text{B.97})$$

$$\hat{Y}_{J_2 m_2(jj) i}^{\rho_2} \left(\hat{Y}_{J m}^{(jj)} \right)^\dagger = \sum_{\rho_1, J_1, m_1} \hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{J m(jj)} \left(\hat{Y}_{J_1 m_1(jj) i}^{\rho_1} \right)^\dagger. \quad (\text{B.98})$$

Finally, the relation

$$\hat{\mathcal{D}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2}^{J m(jj)} = (-1)^{\tilde{Q}_1+\tilde{Q}_2-J} \hat{\mathcal{D}}_{J_2 m_2(jj) \rho_2 J_1 m_1(jj) \rho_1}^{J m(jj)} \quad (\text{B.99})$$

holds from the first equality in (B.13).

B.7.2 $\hat{\mathcal{E}}$

We plug (B.85) into (3.20) and note that

$$\sum_{i', j', k'=1}^3 \epsilon_{i' j' k'} V_{i' n_1} V_{j' n_2} V_{k' n_3} = \epsilon_{n_1 n_2 n_3} \det V = -i \epsilon_{n_1 n_2 n_3} \quad (\text{B.100})$$

with $\epsilon_{n_1=1 n_2=0 n_3=-1} = +1$ and

$$\hat{\mathcal{Y}}_{Q_1 m_1, \tilde{Q}_1(jj)}^{1 n_1} = (-1)^{Q_1-\tilde{Q}_1+1+m_1+n_1} \left(\hat{\mathcal{Y}}_{Q_1-m_1, \tilde{Q}_1(jj)}^{1-n_1} \right)^\dagger, \quad (\text{B.101})$$

¹⁵See, e.g., eq. (1) in Chapter 10.9.1 of [58].

so that the $\hat{\mathcal{E}}$ coefficient is expressed as

$$\begin{aligned} \hat{\mathcal{E}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2 J_3 m_3(jj) \rho_3} &= i^{\rho_1 + \rho_2 + \rho_3 - 1} \sum_{n_1, n_2, n_3 = -1}^1 \epsilon_{-n_1 n_2 n_3} (-1)^{Q_1 - \tilde{Q}_1 + 1 + m_1 - n_1} \\ &\times \frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{Q_1 - m_1, \tilde{Q}_1(jj)}^{1 n_1} \right)^\dagger \hat{\mathcal{Y}}_{Q_2 m_2, \tilde{Q}_2(jj)}^{1 n_2} \hat{\mathcal{Y}}_{Q_3 m_3, \tilde{Q}_3(jj)}^{1 n_3} \right\}. \end{aligned} \quad (\text{B.102})$$

Here, it can be seen from the formulas in section B.5 that

$$\epsilon_{-n_1 n_2 n_3} (-1)^{-n_1} = -\sqrt{2} C_{1 n_2 1 n_3}^{1 n_1} \quad (\text{B.103})$$

holds. This and (B.82) lead to

$$\begin{aligned} \hat{\mathcal{E}}_{J' m_1(jj) \rho_1 J'' m_2(jj) \rho_2 J_3 m_3(jj) \rho_3} &= i^{\rho_1 + \rho_2 + \rho_3 - 1} (-1)^{Q_1 + m_1 + 2j} \\ &\times \sqrt{6n(2\tilde{Q}_1 + 1)(2Q_2 + 1)(2\tilde{Q}_2 + 1)(2Q_3 + 1)(2\tilde{Q}_3 + 1)} \\ &\times \begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & 1 \\ Q_3 & \tilde{Q}_3 & 1 \end{Bmatrix} C_{Q_2 m_2 Q_3 m_3}^{Q_1 - m_1} \begin{Bmatrix} \tilde{Q}_1 & \tilde{Q}_2 & \tilde{Q}_3 \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{B.104})$$

Finally, we use

$$C_{Q_2 m_2 Q_3 m_3}^{Q_1 - m_1} = (-1)^{-Q_2 + Q_3 + m_1} \sqrt{2Q_1 + 1} \begin{pmatrix} Q_1 & Q_2 & Q_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \quad (\text{B.105})$$

which follows from (B.52), and note $m_1 \in \mathbb{Z}$,

$$\begin{aligned} -Q_1 - \tilde{Q}_1 - \rho_1 &= -2J_1 - \delta_{\rho_1, 1} - \delta_{\rho_1, -1} - \rho_1 \in 2\mathbb{Z}, \\ Q_2 - \tilde{Q}_2 - \rho_2 &= \delta_{\rho_2, 1} - \delta_{\rho_2, -1} - \rho_2 = 0, \\ -Q_3 - \tilde{Q}_3 - \rho_3 &\in 2\mathbb{Z} \end{aligned} \quad (\text{B.106})$$

and (B.92), to arrive at the formula

$$\begin{aligned} \hat{\mathcal{E}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2 J_3 m_3(jj) \rho_3} &= i^{-\rho_1 - \rho_2 - \rho_3 - 1} (-1)^{-\tilde{Q}_1 - \tilde{Q}_2 - \tilde{Q}_3 + 2j} \\ &\times \sqrt{6n(2J_1 + 1)(2J_1 + 2\rho_1^2 + 1)(2J_2 + 1)(2J_2 + 2\rho_2^2 + 1)(2J_3 + 1)(2J_3 + 2\rho_3^2 + 1)} \\ &\times \begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & 1 \\ Q_3 & \tilde{Q}_3 & 1 \end{Bmatrix} \begin{pmatrix} Q_1 & Q_2 & Q_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} \tilde{Q}_1 & \tilde{Q}_2 & \tilde{Q}_3 \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{B.107})$$

Equivalently, we can rewrite (3.20) as

$$\begin{aligned} &\sum_{i', j' = 1}^3 \epsilon_{i' j' k'} \hat{Y}_{J_1 m_1(jj) i'}^{\rho_1} \hat{Y}_{J_2 m_2(jj) j'}^{\rho_2} \\ &= \sum_{\rho_3, J_3, m_3} \hat{\mathcal{E}}_{J_1 m_1(jj) \rho_1 J_2 m_2(jj) \rho_2 J_3 m_3(jj) \rho_3} \left(\hat{Y}_{J_3 m_3(jj) k'}^{\rho_3} \right)^\dagger. \end{aligned} \quad (\text{B.108})$$

B.7.3 $\hat{\mathcal{F}}$

In the expression

$$\hat{\mathcal{F}}_{J_2 m_2 (jj) \kappa_2 J m (jj)}^{J_1 m_1 (jj) \kappa_1} = \sum_{\alpha_1 = -\frac{1}{2}}^{\frac{1}{2}} \frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{U_1 m_1, \tilde{U}_1 (jj)}^{\frac{1}{2} \alpha_1} \right)^\dagger \hat{\mathcal{Y}}_{U_2 m_2, \tilde{U}_2 (jj)}^{\frac{1}{2} \alpha_1} \hat{\mathcal{Y}}_{J m, J (jj)}^{00} \right\} \quad (\text{B.109})$$

with $U_a = J_a + \frac{1}{2} \delta_{\kappa_a, 1}$, $\tilde{U}_a = J_a + \frac{1}{2} \delta_{\kappa_a, -1}$ ($a = 1, 2$), which is obtained by plugging (B.85) and (B.86) into (3.21), we insert $1 = \sum_{\alpha_2 = -\frac{1}{2}}^{\frac{1}{2}} \delta_{\alpha_1 \alpha_2} = \sum_{\alpha_2 = -\frac{1}{2}}^{\frac{1}{2}} C_{\frac{1}{2} \alpha_2 00}^{\frac{1}{2} \alpha_1}$ and apply the formula (B.82). Then, the $\hat{\mathcal{F}}$ coefficient becomes

$$\begin{aligned} \hat{\mathcal{F}}_{J_2 m_2 (jj) \kappa_2 J m (jj)}^{J_1 m_1 (jj) \kappa_1} &= (-1)^{\tilde{U}_1 + 2j} \sqrt{2n(2\tilde{U}_1 + 1)(2U_2 + 1)(2\tilde{U}_2 + 1)(2J + 1)} \\ &\quad \times \begin{Bmatrix} U_1 & \tilde{U}_1 & \frac{1}{2} \\ U_2 & \tilde{U}_2 & \frac{1}{2} \\ J & J & 0 \end{Bmatrix} C_{U_2 m_2 J m}^{U_1 m_1} \begin{Bmatrix} \tilde{U}_1 & \tilde{U}_2 & J \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{B.110})$$

Finally, noting

$$(2U_2 + 1)(2\tilde{U}_2 + 1) = (2J'' + 1)(2J'' + 2) \quad (\text{B.111})$$

and (B.94) reduces (B.110) to

$$\begin{aligned} \hat{\mathcal{F}}_{J_2 m_2 (jj) \kappa_2 J m (jj)}^{J_1 m_1 (jj) \kappa_1} &= (-1)^{2j + U_2 + J + \frac{1}{2}} \sqrt{n(2\tilde{U}_1 + 1)(2J_2 + 1)(2J_2 + 2)(2J + 1)} \\ &\quad \times C_{U_2 m_2 J m}^{U_1 m_1} \begin{Bmatrix} U_1 & \tilde{U}_1 & \frac{1}{2} \\ \tilde{U}_2 & U_2 & J \end{Bmatrix} \begin{Bmatrix} \tilde{U}_1 & \tilde{U}_2 & J \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{B.112})$$

More explicitly, for $(\kappa_1, \kappa_2) = (1, 1)$,

$$\begin{aligned} \hat{\mathcal{F}}_{J_2 m_2 (jj) 1 J m (jj)}^{J_1 m_1 (jj) 1} &= (-1)^{2j + J_1 + J_2 + \frac{1}{2} - m_2} \sqrt{n(J + J_1 + J_2 + 2)(-J + J_1 + J_2 + 1)} \\ &\quad \times C_{J_1 + \frac{1}{2} m_1 J_2 + \frac{1}{2} - m_2}^{J m} \begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix}, \end{aligned} \quad (\text{B.113})$$

for $(\kappa_1, \kappa_2) = (-1, -1)$,

$$\begin{aligned} \hat{\mathcal{F}}_{J_2 m_2 (jj) -1 J m (jj)}^{J_1 m_1 (jj) -1} &= (-1)^{2j + J_1 + J_2 + \frac{1}{2} - m_2} \sqrt{n(J + J_1 + J_2 + 2)(-J + J_1 + J_2 + 1)} \\ &\quad \times C_{J_1 m_1 J_2 - m_2}^{J m} \begin{Bmatrix} J_1 + \frac{1}{2} & J_2 + \frac{1}{2} & J \\ j & j & j \end{Bmatrix}, \end{aligned} \quad (\text{B.114})$$

for $(\kappa_1, \kappa_2) = (1, -1)$,

$$\hat{\mathcal{F}}_{J_2 m_2 (jj) -1 J m (jj)}^{J_1 m_1 (jj) 1} = (-1)^{2j + J_1 + J_2 - m_2} \sqrt{n \left(J + J_1 - J_2 + \frac{1}{2} \right) \left(J - J_1 + J_2 + \frac{1}{2} \right)}$$

$$\times C_{J_1 + \frac{1}{2} \ m_1 \ J_2 \ -m_2}^{J \ m} \begin{Bmatrix} J_1 & J_2 + \frac{1}{2} & J \\ j & j & j \end{Bmatrix}, \quad (\text{B.115})$$

and, for $(\kappa_1, \kappa_2) = (-1, 1)$,

$$\begin{aligned} \hat{\mathcal{F}}_{J_2 \ m_2 \ (jj) \ 1 \ J \ m \ (jj)}^{J_1 \ m_1 \ (jj) \ -1} &= (-1)^{2j+J_1+J_2-m_2} \sqrt{n \left(J + J_1 - J_2 + \frac{1}{2} \right) \left(J - J_1 + J_2 + \frac{1}{2} \right)} \\ &\times C_{J_1 \ m_1 \ J_2 + \frac{1}{2} \ -m_2}^{J \ m} \begin{Bmatrix} J_1 + \frac{1}{2} & J_2 & J \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{B.116})$$

The property

$$C_{U_2 \ m_2 \ J \ m}^{U_1 \ m_1} = (-1)^{J+m} \sqrt{\frac{2U_1+1}{2U_2+1}} C_{U_1 \ -m_1 \ J \ m}^{U_2 \ -m_2} \quad (\text{B.117})$$

together with (B.22) and (B.93) leads to

$$\hat{\mathcal{F}}_{J_2 \ m_2 \ (jj) \ \kappa_2 \ J \ m \ (jj)}^{J_1 \ m_1 \ (jj) \ \kappa_1} = (-1)^{J+\tilde{U}_1-\tilde{U}_2+m_1-m_2-\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2} \hat{\mathcal{F}}_{J_1 \ -m_1 \ (jj) \ \kappa_1 \ J \ m \ (jj)}^{J_2 \ -m_2 \ (jj) \ \kappa_2}. \quad (\text{B.118})$$

B.7.4 $\hat{\mathcal{G}}$

We substitute (B.85) and (B.86) in (3.22) to obtain

$$\begin{aligned} \hat{\mathcal{G}}_{J_2 \ m_2 \ (jj) \ \kappa_2 \ J \ m \ (jj) \ \rho}^{J_1 \ m_1 \ (jj) \ \kappa_1} &= i^\rho \sum_{\alpha, \beta = -\frac{1}{2}}^{\frac{1}{2}} \sum_{n'=-1}^1 \left(\sum_{i=1}^3 \sigma_{\alpha\beta}^i V_{i \ n'} \right) \\ &\times \frac{1}{n} \text{tr}_n \left\{ \left(\hat{\mathcal{Y}}_{U_1 \ m_1, \tilde{U}_1 \ (jj)}^{\frac{1}{2} \ \alpha} \right)^\dagger \hat{\mathcal{Y}}_{U_2 \ m_2, \tilde{U}_2 \ (jj)}^{\frac{1}{2} \ \beta} \hat{\mathcal{Y}}_{Q \ m, \tilde{Q} \ (jj)}^{1 \ n'} \right\}. \end{aligned} \quad (\text{B.119})$$

Note that

$$\sum_{i=1}^3 \sigma_{\alpha\beta}^i V_{i \ n'} = \sqrt{3} C_{\frac{1}{2} \ \beta \ 1 \ n'}^{\frac{1}{2} \ \alpha} \quad (\text{B.120})$$

holds from

$$\begin{aligned} \sum_{i=1}^3 \sigma_{\alpha\beta}^i V_{i \ n'} &= -\frac{n'}{\sqrt{2}} \left(\delta_{\alpha, \frac{1}{2}} \delta_{\beta, -\frac{1}{2}} + \delta_{\alpha, -\frac{1}{2}} \delta_{\beta, \frac{1}{2}} \right) - \frac{|n'|}{\sqrt{2}} \left(\delta_{\alpha, \frac{1}{2}} \delta_{\beta, -\frac{1}{2}} - \delta_{\alpha, -\frac{1}{2}} \delta_{\beta, \frac{1}{2}} \right) \\ &+ \delta_{n', 0} \left(\delta_{\alpha, \frac{1}{2}} \delta_{\beta, \frac{1}{2}} - \delta_{\alpha, -\frac{1}{2}} \delta_{\beta, -\frac{1}{2}} \right), \end{aligned} \quad (\text{B.121})$$

and formulas in section B.5. Together with this, (B.82), (B.92), and (B.111) lead to

$$\begin{aligned} \hat{\mathcal{G}}_{J_2 \ m_2 \ (jj) \ \kappa_2 \ J \ m \ (jj) \ \rho}^{J_1 \ m_1 \ (jj) \ \kappa_1} &= i^\rho (-1)^{\tilde{U}_1+2j} \sqrt{6n(2\tilde{U}_1+1)(2J_2+1)(2J_2+2)(2J+1)(2J+2\rho^2+1)} \\ &\times \begin{Bmatrix} U_1 & \tilde{U}_1 & \frac{1}{2} \\ U_2 & \tilde{U}_2 & \frac{1}{2} \\ Q & \tilde{Q} & 1 \end{Bmatrix} C_{U_2 \ m_2 \ Q \ m}^{U_1 \ m_1} \begin{Bmatrix} \tilde{U}_1 & \tilde{U}_2 & \tilde{Q} \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{B.122})$$

C Computational details of one-point functions

In this appendix, the details of the computation of (5.9), (5.11), and (5.13) are presented. The gauge transformation property of the one-point function $\langle y_{Jm,\rho}(p) \rangle$ is also discussed.

C.1 (5.9)

Here, we calculate the sum of $\hat{\mathcal{E}}$ with respect to m' in (5.9). From the expression of $\hat{\mathcal{E}}$ (B.107) and $2\tilde{Q}' = 2(J' + \delta_{\rho', -1}) \in 2\mathbb{Z}$, it reduces to the sum of $3j$ symbols:

$$\begin{aligned} & \sum_{m'=-Q'}^{Q'} (-1)^{-m'} \hat{\mathcal{E}}_{\bar{J}-\bar{m}(jj) \bar{\rho} J' m' (jj) \rho' J' -m' (jj) \rho'} = i^{-\bar{\rho}-2\rho'-1} (-1)^{-\tilde{Q}+2j} \\ & \times \sqrt{6n(2\bar{J}+1)(2\bar{J}+2\bar{\rho}^2+1)(2J'+1)(2J'+2\rho'^2+1)} \\ & \times \left\{ \begin{matrix} \bar{Q} & \tilde{Q} & 1 \\ Q' & \tilde{Q}' & 1 \\ Q' & \tilde{Q}' & 1 \end{matrix} \right\} \left\{ \begin{matrix} \tilde{Q} & \tilde{Q}' & \tilde{Q}' \\ j & j & j \end{matrix} \right\} \sum_{m'=-Q'}^{Q'} (-1)^{-m'} \begin{pmatrix} \bar{Q} & Q' & Q' \\ -\bar{m} & m' & -m' \end{pmatrix} \end{aligned} \quad (\text{C.1})$$

with $\tilde{Q} = \bar{J} + \delta_{\bar{\rho}, -1}$.

The relation ¹⁶

$$\sum_{m'=-Q'}^{Q'} (-1)^{-m'} \begin{pmatrix} \bar{Q} & Q' & Q' \\ -\bar{m} & m' & -m' \end{pmatrix} = \sum_{m'=-Q'}^{Q'} (-1)^{-m'+\bar{Q}} \sum_{J=|\bar{Q}-Q'|}^{\bar{Q}+Q'} \sum_{m=-J}^J C_{\bar{Q}-\bar{m} Q' m'}^{Jm} C_{Jm Q' -m'}^{00} \quad (\text{C.2})$$

together with

$$\begin{aligned} C_{\bar{Q}-\bar{m} Q' m'}^{Jm} &= (-1)^{Q'+m'} \sqrt{\frac{2J+1}{2\bar{Q}+1}} C_{J-m Q' m'}^{\bar{Q}\bar{m}}, \\ C_{J-m Q' m'}^{\bar{Q}\bar{m}} &= (-1)^{J+Q'-\bar{Q}} C_{Jm Q' -m'}^{\bar{Q}-\bar{m}} \end{aligned} \quad (\text{C.3})$$

and the orthogonality (B.17) leads to

$$\begin{aligned} \sum_{m'=-Q'}^{Q'} (-1)^{-m'} \begin{pmatrix} \bar{Q} & Q' & Q' \\ -\bar{m} & m' & -m' \end{pmatrix} &= (-1)^{Q'} \sqrt{2Q'+1} \delta_{\bar{Q}0} \delta_{\bar{m}0} \\ &= (-1)^{Q'} \sqrt{2Q'+1} \delta_{\bar{\rho}, -1} \delta_{\bar{J}0} \delta_{\bar{m}0}. \end{aligned} \quad (\text{C.4})$$

In addition,

$$\left\{ \begin{matrix} 1 & \tilde{Q}' & \tilde{Q}' \\ j & j & j \end{matrix} \right\} = \left\{ \begin{matrix} j & \tilde{Q}' & j \\ 1 & j & \tilde{Q}' \end{matrix} \right\} = (-1)^{\tilde{Q}'+2j+1} \frac{1}{2\sqrt{j(j+1)n}} \sqrt{\frac{\tilde{Q}'(\tilde{Q}'+1)}{2\tilde{Q}'+1}} \quad (\text{C.5})$$

¹⁶See, e.g., eq. (13) in Chapter 8.1.3 of [58].

and

$$\begin{Bmatrix} 0 & 1 & 1 \\ Q' & \tilde{Q}' & 1 \\ Q' & \tilde{Q}' & 1 \end{Bmatrix} = \frac{(-1)^{Q'+\tilde{Q}'}}{\sqrt{3(2Q'+1)}} \begin{Bmatrix} 1 & 1 & 1 \\ \tilde{Q}' & \tilde{Q}' & Q' \end{Bmatrix} = \frac{-Q'(Q'+1) + \tilde{Q}'(\tilde{Q}'+1) + 2}{6\sqrt{2(2Q'+1)\tilde{Q}'(\tilde{Q}'+1)(2\tilde{Q}'+1)}}, \quad (\text{C.6})$$

which are seen from, e.g., Table 9.2 in Chapter 9.12 and eq. (2) in Chapter 10.9.1 of [58]. Plugging these into (C.1), we end up with

$$\begin{aligned} & \sum_{m'=-Q'}^{Q'} (-1)^{-m'} \hat{\mathcal{E}}_{\bar{J}-\bar{m}(jj) \bar{\rho} J' m' (jj) \rho' J' -m' (jj) \rho'} \\ &= \frac{-1}{2\sqrt{j(j+1)}} (2J'+1 + 2\delta_{\rho',1}) [\rho'(J'+1) - 1] \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0}. \end{aligned} \quad (\text{C.7})$$

Here, we have used

$$\begin{aligned} 2j &\in \mathbb{Z}, \quad Q' + \tilde{Q}' + \rho' = 2(J' + \delta_{\rho',1}) \in 2\mathbb{Z}, \\ -Q'(Q'+1) + \tilde{Q}'(\tilde{Q}'+1) + 2 &= -2[\rho'(J'+1) - 1]. \end{aligned} \quad (\text{C.8})$$

C.2 (5.11)

Let us calculate the sum of $\hat{\mathcal{D}}$ with respect to m in (5.11).

For $9j$ symbols, each elementary permutation of rows or columns gives a multiplicative sign factor,

$$(-1)^{\text{sum of all the angular momenta}}, \quad (\text{C.9})$$

as seen from Chapter 10.4.1 of [58]. Namely,

$$\begin{aligned} \begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{Bmatrix} &= (-1)^{a+b+\dots+k} \begin{Bmatrix} d & e & f \\ a & b & c \\ g & h & k \end{Bmatrix} = (-1)^{a+b+\dots+k} \begin{Bmatrix} a & b & c \\ g & h & k \\ d & e & f \end{Bmatrix} \\ &= (-1)^{a+b+\dots+k} \begin{Bmatrix} b & a & c \\ e & d & f \\ h & g & k \end{Bmatrix} = (-1)^{a+b+\dots+k} \begin{Bmatrix} a & c & b \\ d & f & e \\ g & k & h \end{Bmatrix}. \end{aligned} \quad (\text{C.10})$$

From (B.99),

$$\sum_{m=-J}^J \left(\hat{\mathcal{D}}_{\bar{J}-\bar{m}(jj) \bar{\rho} J m (jj) 0}^{J m (jj)} - \hat{\mathcal{D}}_{J m (jj) 0 \bar{J}-\bar{m}(jj) \bar{\rho}}^{J m (jj)} \right) = \left(1 - (-1)^{\tilde{Q}} \right) \sum_{m=-J}^J \hat{\mathcal{D}}_{\bar{J}-\bar{m}(jj) \bar{\rho} J m (jj) 0}^{J m (jj)}$$

$$\begin{aligned}
&= \left(1 - (-1)^{\tilde{Q}}\right) i^{\bar{\rho}} (-1)^{J+2j+1} \sqrt{3n(2\bar{J}+1)(2\bar{J}+2\bar{\rho}^2+1)} (2J+1)^{3/2} \\
&\times \begin{Bmatrix} \bar{Q} & \tilde{\bar{Q}} & 1 \\ J & J & 1 \\ J & J & 0 \end{Bmatrix} \begin{Bmatrix} J & \tilde{\bar{Q}} & J \\ j & j & j \end{Bmatrix} \left(\sum_{m=-J}^J C_{\bar{Q}-\bar{m}Jm}^{Jm} \right). \tag{C.11}
\end{aligned}$$

Because the sum of the C-G coefficient in the last line can be written as the sum of a $3j$ symbol,

$$\sum_{m=-J}^J C_{\bar{Q}-\bar{m}Jm}^{Jm} = \sum_{m=-J}^J (-1)^{\bar{Q}-J+m} \sqrt{2J+1} \begin{Bmatrix} \bar{Q} & J & J \\ -\bar{m} & m & -m \end{Bmatrix}, \tag{C.12}$$

it can be computed similarly to the derivation of (C.4):

$$\sum_{m=-J}^J C_{\bar{Q}-\bar{m}Jm}^{Jm} = (2J+1) \delta_{\bar{Q}0} \delta_{\bar{m}0} = (2J+1) \delta_{\bar{\rho},-1} \delta_{J0} \delta_{\bar{m}0}. \tag{C.13}$$

Then, (C.11) reduces to

$$\begin{aligned}
&\sum_{m=-J}^J \left(\hat{\mathcal{D}}_{\bar{J}-\bar{m}(jj)\bar{\rho}Jm(jj)0}^{Jm(jj)} - \hat{\mathcal{D}}_{Jm(jj)0\bar{J}-\bar{m}(jj)\bar{\rho}}^{Jm(jj)} \right) \\
&= -6i(-1)^{J+2j+1} \sqrt{n} (2J+1)^{5/2} \delta_{\bar{\rho},-1} \delta_{J0} \delta_{\bar{m}0} \begin{Bmatrix} 0 & 1 & 1 \\ J & J & 1 \\ J & J & 0 \end{Bmatrix} \begin{Bmatrix} J & 1 & J \\ j & j & j \end{Bmatrix}. \tag{C.14}
\end{aligned}$$

As in (C.5), we have

$$\begin{Bmatrix} J & 1 & J \\ j & j & j \end{Bmatrix} = (-1)^{J+2j+1} \frac{1}{2\sqrt{j(j+1)n}} \sqrt{\frac{J(J+1)}{2J+1}}. \tag{C.15}$$

Equation (C.10) and the formula ¹⁷

$$\begin{Bmatrix} a & b & c \\ d & 0 & f \\ g & h & 0 \end{Bmatrix} = \delta_{df} \delta_{bh} \delta_{cf} \delta_{gh} \frac{(-1)^{a-b-c}}{(2b+1)(2c+1)} \tag{C.16}$$

lead to

$$\begin{Bmatrix} 0 & 1 & 1 \\ J & J & 1 \\ J & J & 0 \end{Bmatrix} = - \begin{Bmatrix} 1 & 0 & 1 \\ J & J & 1 \\ J & J & 0 \end{Bmatrix} = \begin{Bmatrix} J & J & 1 \\ 1 & 0 & 1 \\ J & J & 0 \end{Bmatrix} = -\frac{1}{3(2J+1)}. \tag{C.17}$$

¹⁷See, e.g., eq. (3) in Chapter 10.9.1 of [58].

Plugging (C.15) and (C.17) into (C.14), we end up with

$$\sum_{m=-J}^J \left(\hat{\mathcal{D}}_{\bar{J}-\bar{m}(jj)\bar{\rho}}^{Jm(jj)} - \hat{\mathcal{D}}_{Jm(jj)0\bar{J}-\bar{m}(jj)\bar{\rho}}^{Jm(jj)} \right) = i \frac{\sqrt{J(J+1)}(2J+1)}{\sqrt{j(j+1)}} \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0}. \quad (\text{C.18})$$

C.3 (5.13)

Here, we compute the sum of $\hat{\mathcal{G}}$ with respect to m in (5.13). From the expression of $\hat{\mathcal{G}}$ (B.122), the sum reduces to the sum of the C-G coefficient $C_{U-m\bar{Q}-\bar{m}}^{U-m}$. It can be obtained by using (C.13) as

$$\sum_{m=-U}^U C_{U-m\bar{Q}-\bar{m}}^{U-m} = (-1)^{\bar{Q}} \sum_{m=-U}^U C_{\bar{Q}-\bar{m}U-m}^{U-m} = (-1)^{\bar{Q}} (2U+1) \delta_{\bar{Q}0} \delta_{\bar{m}0} = (2U+1) \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0}. \quad (\text{C.19})$$

Then, we have

$$\begin{aligned} \sum_{m=-U}^U \hat{\mathcal{G}}_{J-m(jj)\kappa\bar{J}-\bar{m}(jj)\bar{\rho}}^{J-m(jj)\kappa} &= -i(-1)^{\tilde{U}+2j} 3\sqrt{2n(2\tilde{U}+1)(2J+1)(2J+2)(2U+1)} \\ &\quad \times \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0} \begin{Bmatrix} U & \tilde{U} & \frac{1}{2} \\ U & \tilde{U} & \frac{1}{2} \\ 0 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} \tilde{U} & \tilde{U} & 1 \\ j & j & j \end{Bmatrix}. \end{aligned} \quad (\text{C.20})$$

By a similar derivation of (C.5) and (C.6),

$$\begin{aligned} \begin{Bmatrix} \tilde{U} & \tilde{U} & 1 \\ j & j & j \end{Bmatrix} &= (-1)^{\tilde{U}+2j+1} \frac{1}{2\sqrt{j(j+1)n}} \sqrt{\frac{\tilde{U}(\tilde{U}+1)}{2\tilde{U}+1}}, \\ \begin{Bmatrix} U & \tilde{U} & \frac{1}{2} \\ U & \tilde{U} & \frac{1}{2} \\ 0 & 1 & 1 \end{Bmatrix} &= \frac{(-1)^{U+\tilde{U}+\frac{3}{2}}}{\sqrt{3(2U+1)}} \begin{Bmatrix} \frac{1}{2} & \tilde{U} & U \\ \tilde{U} & \frac{1}{2} & 1 \end{Bmatrix} = \frac{\frac{3}{4} - U(U+1) + \tilde{U}(\tilde{U}+1)}{3\sqrt{2\tilde{U}(\tilde{U}+1)(2U+1)(2\tilde{U}+1)}}. \end{aligned} \quad (\text{C.21})$$

$$\quad (\text{C.22})$$

Note that $U + \tilde{U} + \frac{3}{2} = 2J + 2 \in \mathbb{Z}$ since J is an integer or a half-integer.

Plugging (C.21) and (C.22) into (C.20), we end up with

$$\begin{aligned} \sum_{m=-U}^U \hat{\mathcal{G}}_{J-m(jj)\kappa\bar{J}-\bar{m}(jj)\bar{\rho}}^{J-m(jj)\kappa} &= i \frac{1}{2\sqrt{j(j+1)}} \sqrt{\frac{2U+1}{2\tilde{U}+1}} \sqrt{(2J+1)(2J+2)} \delta_{\bar{\rho},-1} \delta_{\bar{J}0} \delta_{\bar{m}0} \\ &\quad \times \left[\frac{3}{4} - U(U+1) + \tilde{U}(\tilde{U}+1) \right] \end{aligned}$$

$$= i \frac{1}{2\sqrt{j(j+1)}} (2U+1) \left[\frac{3}{4} - \kappa \left(J + \frac{3}{4} \right) \right] \delta_{\bar{\rho}, -1} \delta_{\bar{J}0} \delta_{\bar{m}0}. \quad (\text{C.23})$$

C.4 Gauge transformation property of $y_{Jm,\rho}(p)$

Here, we consider the gauge transformation of $y_{Jm,\rho}(p)$. The gauge transformations of $\tilde{X}_{8,9,10}$ are

$$\begin{aligned} \delta \tilde{X}_9 &= i \frac{M}{3} [L_1, \Omega] + i [\tilde{X}_9, \Omega], \\ \delta \tilde{X}_{10} &= i \frac{M}{3} [L_2, \Omega] + i [\tilde{X}_{10}, \Omega], \\ \delta \tilde{X}_8 &= i \frac{M}{3} [L_3, \Omega] + i [\tilde{X}_8, \Omega], \end{aligned} \quad (\text{C.24})$$

where Ω is a gauge transformation parameter. Then the gauge transformation of $\vec{Y} = (\tilde{X}_9, \tilde{X}_{10}, \tilde{X}_8)^T$ is expressed as

$$\delta \vec{Y}(p) = i \frac{M}{3} \vec{L} \circ \tilde{\Omega}(p) + i \int \frac{d^2 q}{(2\pi)^2} [\vec{Y}(q), \tilde{\Omega}(p-q)] \quad (\text{C.25})$$

in the momentum representation in \mathbb{R}^2 . Furthermore, in terms of modes on fuzzy S^2 ,

$$\vec{Y}(p) = \sum_{\rho=-1}^1 \sum_{J=\delta_{\rho,0}}^{2j-\delta_{\rho,-1}} \sum_{m=-Q}^Q \vec{Y}_{Jm}^{\rho}(jj) \otimes y_{Jm,\rho}(p), \quad (\text{C.26})$$

$$\tilde{\Omega}(p) = \sum_{J=0}^{2j} \sum_{m=-J}^J \hat{Y}_{Jm}^{(jj)} \otimes \omega_{Jm}(p), \quad (\text{C.27})$$

(C.25) reads

$$\begin{aligned} \delta y_{Jm,\rho}(p) &= \delta_{\rho,0} i \frac{M}{3} \sqrt{J(J+1)} \omega_{Jm}(p) \\ &+ \sum_{\rho'=-1}^1 \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{m=-Q'}^{Q'} \sum_{J''=0}^{2j} \sum_{m''=-J''}^{J''} (-1)^{m''-m+1} \hat{\mathcal{D}}_{J-m}^{J''-m''(jj)}{}_{\rho J' m' (jj) \rho'} \\ &\times i \int \frac{d^2 q}{(2\pi)^2} \left[y_{J' m', \rho'}(q) \omega_{J'' m''}(p-q) - (-1)^{\tilde{Q}'+\tilde{Q}-J''} \omega_{J'' m''}(p-q) y_{J' m', \rho'}(q) \right], \end{aligned} \quad (\text{C.28})$$

where we have used (B.99).

Let us focus on the case of $J = m = 0$ where the first term of (C.28) vanishes:

$$\begin{aligned} \delta y_{00,\rho}(p) &= \sum_{\rho'=-1}^1 \sum_{J'=\delta_{\rho',0}}^{2j-\delta_{\rho',-1}} \sum_{m=-Q'}^{Q'} \sum_{J''=0}^{2j} \sum_{m''=-J''}^{J''} (-1)^{m''+1} \hat{\mathcal{D}}_{00(jj)\rho J' m' (jj)\rho'}^{J''-m''(jj)} \\ &\times i \int \frac{d^2 q}{(2\pi)^2} \left[y_{J' m', \rho'}(q) \omega_{J'' m''}(p-q) - (-1)^{\tilde{Q}' + \delta_{\rho, -1} - J''} \omega_{J'' m''}(p-q) y_{J' m', \rho'}(q) \right]. \end{aligned} \quad (\text{C.29})$$

Since $y_{00,\rho=0}(p)$ does not exist, we consider the cases $\rho = \pm 1$. Note that $\text{tr}_k y_{00,\rho}(p)$ is gauge invariant when each term of the summand satisfies

$$\tilde{Q}' + \delta_{\rho, -1} - J'' = J' - J'' + \delta_{\rho, -1} + \delta_{\rho', -1} \in 2\mathbb{Z}. \quad (\text{C.30})$$

For $\rho = 1$, $\hat{\mathcal{D}}$ can be computed as

$$\begin{aligned} \hat{\mathcal{D}}_{00(jj)1 J' m' (jj)\rho'}^{J''-m''(jj)} &= \delta_{\rho',0} \left[i \frac{m'}{\sqrt{J'(J'+1)}} \delta_{J' J''} \delta_{m'+m'',0} \right] \\ &+ \delta_{\rho',1} \left[-\sqrt{\frac{(J'-m'+1)(J'+m'+1)}{(J'+1)(2J'+1)}} \delta_{J' J''} \delta_{m'+m'',0} \right] \\ &+ \delta_{\rho',-1} \left[-\sqrt{\frac{(J'-m'+1)(J'+m'+1)}{(J'+1)(2J'+3)}} \delta_{J'+1, J''} \delta_{m'+m'',0} \right], \end{aligned} \quad (\text{C.31})$$

from which we find that (C.30) is satisfied for each case of $\rho' = 0, \pm 1$. Hence, $\text{tr}_k y_{00,\rho=1}(p)$ is gauge invariant.

On the other hand, for $\rho = -1$, $\hat{\mathcal{D}}$ becomes

$$\begin{aligned} \hat{\mathcal{D}}_{00(jj)-1 J' m' (jj)\rho'}^{J''-m''(jj)} &= \delta_{\rho',0} \left[\frac{i}{2\sqrt{j(j+1)}} \sqrt{J'(J'+1)} \delta_{J' J''} \delta_{m'+m'',0} \right] \\ &+ \delta_{\rho',1} \left[\frac{-1}{2\sqrt{j(j+1)}} \sqrt{\frac{(J'+2j+2)(2j-J')(J'+1)}{2J'+3}} \delta_{J'+1, J''} \delta_{m'+m'',0} \right] \\ &+ \delta_{\rho',-1} \left[\frac{-1}{2\sqrt{j(j+1)}} \sqrt{\frac{(J'+2j+2)(2j-J')(J'+1)}{2J'+1}} \delta_{J' J''} \delta_{m'+m'',0} \right]. \end{aligned} \quad (\text{C.32})$$

Thus, (C.30) is satisfied for $\rho' = \pm 1$, but not for $\rho' = 0$. This shows that $\text{tr}_k y_{00,\rho=-1}(p)$ is not gauge invariant.

C.5 $\text{tr}_n \vec{Y}_{00(jj)}^{\rho=-1}$

Here, we show that $\text{tr}_n \vec{Y}_{00(jj)}^{\rho=-1}$ vanishes, which is consistent with the gauge invariance.

From the definition of vector and scalar fuzzy spherical harmonics,

$$\vec{Y}_{00(jj)}^{\rho=-1} = \begin{pmatrix} \frac{-i}{\sqrt{6}} \left(\hat{Y}_{11}^{(jj)} - \hat{Y}_{1-1}^{(jj)} \right) \\ \frac{-1}{\sqrt{6}} \left(\hat{Y}_{11}^{(jj)} + \hat{Y}_{1-1}^{(jj)} \right) \\ \frac{i}{\sqrt{3}} \hat{Y}_{10}^{(jj)} \end{pmatrix}, \quad (\text{C.33})$$

$$\text{tr}_n \hat{Y}_{Jm}^{(jj)} = \sqrt{n} \sum_{r=-j}^j (-1)^{-j+r} C_{jrj-r}^{Jm}. \quad (\text{C.34})$$

Due to the angular momentum conservation, (C.34) vanishes for nonzero m :

$$\text{tr}_n \hat{Y}_{11}^{(jj)} = \text{tr}_n \hat{Y}_{1-1}^{(jj)} = 0. \quad (\text{C.35})$$

We note that $C_{jrj-r}^{10} = (-1)^{2j-1} C_{j-rj}^{10}$ and $j-r \in \mathbb{Z}$ to show that

$$\begin{aligned} \text{tr}_n \hat{Y}_{10}^{(jj)} &= \sqrt{n} \sum_{r=-j}^j (-1)^{-j+r} C_{jrj-r}^{10} = \sqrt{n} \sum_{r=-j}^j (-1)^{-j+r} (-1)^{2j-1} C_{j-rj}^{10} \\ &= -\sqrt{n} \sum_{r=-j}^j (-1)^{j-r} C_{jrj-r}^{10} = -\text{tr}_n \hat{Y}_{10}^{(jj)} = 0. \end{aligned} \quad (\text{C.36})$$

Thus, (C.35) and (C.36) lead to

$$\text{tr}_n \vec{Y}_{00(jj)}^{\rho=-1} = 0. \quad (\text{C.37})$$

D Proof of Theorems 1 and 2

In this appendix, we give proofs of Theorems 1 and 2, which are claimed in the text.

D.1 Theorem 1

Here, in order to show Theorem 1, we evaluate $\mathcal{A}_{II}^{\text{UV}}$ in (6.44):

$$\mathcal{A}_{II}^{\text{UV}} = \left(\frac{M}{3} \right)^2 \sum_{J', J'' \in \text{Region II}} n f(J', J'', J) \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 I_{a,b}, \quad (\text{D.1})$$

$$I_{a,b} \equiv \int \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{g(\tilde{p}, \tilde{q})}{M_{a,b} \left(\tilde{P}_i(J'); \tilde{Q}_k(J''); \tilde{q}, \tilde{p} \right)}, \quad (\text{D.2})$$

where $g(\tilde{p}, \tilde{q})$ and $f(J', J'', J)$ are supposed to have properties (6.45) and (6.36). We also assume that the integration $I_{a,b}$ converges. Recall that we are considering the situation of

$\tilde{p}_\mu = \frac{3}{M}p_\mu$ and $J = \frac{3}{M}u$ typically of the order of $\mathcal{O}(M^{-1})$. In Region II defined by (6.41) and (6.42), it is convenient to change the summation variables from J' and J'' to

$$J_+ \equiv J' + J'', \quad \Delta \equiv J' - J''. \quad (\text{D.3})$$

Then, $J_+ \gg \Delta$ for a sufficiently small M . By noting that both J_+ and Δ take even integers or odd integers since $J' = \frac{1}{2}(J_+ + \Delta)$ and $J'' = \frac{1}{2}(J_+ - \Delta)$ should be integers, the summation can be rewritten as

$$\sum_{J', J'' \in \text{Region II}} = \sum_{J_B \leq J_+ \leq 4j}^{\text{even}} \sum_{|\Delta| \leq J}^{\text{even}} + \sum_{J_B \leq J_+ \leq 4j}^{\text{odd}} \sum_{|\Delta| \leq J}^{\text{odd}}. \quad (\text{D.4})$$

From $|\Delta| \leq J$, it can be seen that there is a constant C_1 such that

$$|f(J', J''; J)| \leq C_1 M^{-N_J - N_\Delta} J_+^{N_1 + N_2}. \quad (\text{D.5})$$

In this region, we can use Edmonds' formula, which holds for $a, b, c \gg f, m, n$ ($f, m, n \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$)¹⁸:

$$\left\{ \begin{matrix} a & a+n & f \\ b+m & b & c \end{matrix} \right\} \simeq \frac{(-1)^{a+b+c+f+m}}{\sqrt{(2a+1)(2b+1)}} d_{mn}^f(\beta), \quad (\text{D.6})$$

where

$$\cos \beta = \frac{a(a+1) + b(b+1) - c(c+1)}{2\sqrt{a(a+1)b(b+1)}}, \quad (\text{D.7})$$

and $d_{mn}^f(\beta)$ is a real function related to the Wigner D -function. It is explicitly given as

$$\begin{aligned} d_{mn}^f(\beta) &= (-1)^{f-n} \frac{1}{2^f} \sqrt{\frac{(f+m)!}{(f-m)!(f+n)!(f-n)!}} (1-X)^{-\frac{m-n}{2}} (1+X)^{-\frac{m+n}{2}} \\ &\times \frac{d^{f-m}}{dX^{f-m}} [(1-X)^{f-n}(1+X)^{f+n}] \end{aligned} \quad (\text{D.8})$$

with $X = \cos \beta$ and $0 < \beta < \frac{\pi}{2}$. By putting $a = J''$, $n = \Delta$, $f = J$, $m = 0$ and $b = c = j$ in this formula, we obtain

$$n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \simeq \frac{1}{2J''+1} (d_{0\Delta}^J(\beta))^2 \quad (\text{D.9})$$

and

$$\cos \beta = \frac{1}{2} \sqrt{\frac{J''(J''+1)}{j(j+1)}} = \frac{J_+}{4j} \left(1 + \mathcal{O}\left(\frac{\Delta}{J_+}, \frac{1}{J_+}\right) \right), \quad (\text{D.10})$$

¹⁸See, e.g., eq. (14) in Chapter 9.9.1 of [58].

which means that there is a constant C_2 such that

$$n \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \leq \frac{C_2}{J_+} (d_{0\Delta}^J(\beta_+))^2 \quad \text{with} \quad \cos \beta_+ \equiv \frac{J_+}{4j}. \quad (\text{D.11})$$

Next, let us evaluate the integral $I_{a,b}$ in (D.2). Recall that the polynomials $\tilde{P}_i(J)$ and $\tilde{Q}_k(J)$ in the denominator of the integrand are actually of the form (6.40). Because of $|\tilde{p}| \ll J''$ and $|\Delta| \ll J_+$ in Region II, we see that $\frac{1}{\tilde{q}^2 + \tilde{P}_i(J')}$ and $\frac{1}{(\tilde{q} + \tilde{p})^2 + \tilde{Q}_k(J'')}$ are bounded from the above as

$$\frac{1}{\tilde{q}^2 + \tilde{P}_i(J')} < \frac{c'}{\tilde{q}^2 + J_+^2}, \quad \frac{1}{(\tilde{q} + \tilde{p})^2 + \tilde{Q}_k(J'')} < \frac{c''}{\tilde{q}^2 + J_+^2}, \quad (\text{D.12})$$

where c' and c'' are some constants of the order of $\mathcal{O}(1)$. Therefore, with use of (6.45), $I_{a,b}$ is bounded as

$$|I_{a,b}| \leq C'_3 \int \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{|g(\tilde{p}, \tilde{q})|}{(\tilde{q}^2 + J_+^2)^{a+b}} \leq C_3 M^{-n_p} J_+^{2+n_q-2(a+b)}. \quad (\text{D.13})$$

C'_3 and C_3 are constants of the order of $\mathcal{O}(1)$.

Note that the identity

$$\sum_{\Delta=-J}^J (d_{0\Delta}^J(\beta_+))^2 = 1 \quad (\text{D.14})$$

follows from

$$d_{MM'}^J(\beta) = (-1)^{M-M'} d_{M'M}^J(\beta), \quad (\text{D.15})$$

$$\sum_{M''=-J}^J (-1)^{M''-M'} d_{MM''}^J(\beta) d_{M''M'}^J(\beta) = \delta_{MM'}. \quad (\text{D.16})$$

These are seen, e.g., in eq. (1) in Chapter 4.4 and eq. (10) in Chapter 4.7.2 of [58]. Combining (D.5), (D.11), (D.13) and (D.14), we have

$$|A_{II}^{UV}| = C_{II} M^{2-n_p-N_J-N_\Delta} \sum_{J_B \leq J_+ \leq 4j} J_+^{1+n_q+N_1+N_2-2(a+b)} \quad (\text{D.17})$$

with an $\mathcal{O}(1)$ constant C_{II} .

Recalling that $j = \mathcal{O}(M^{-2})$ and $J_B = \mathcal{O}(M^{-2\alpha})$, we end up with

$$|\mathcal{A}_{II}^{UV}| \leq \begin{cases} C_+ M^{W^+} & (w > 0) \\ C_0 M^{W^-} |\ln M| & (w = 0) \\ C_- M^{-2\alpha w - n_p - N_J - N_\Delta + 2} < C_- M^{W^-} & (w < 0), \end{cases} \quad (\text{D.18})$$

where $w \equiv N_1 + N_2 - 2(a+b) + n_q + 2$, $W^+ \equiv -N_J - N_\Delta - n_p + 2 - 2w$ and $W^- \equiv -N_J - N_\Delta - n_p + 2 - w$. C_+ , C_0 and C_- are M -independent constants. That immediately proves the theorem.

D.2 Theorem 2

In Region I, we express $\mathcal{A}_I^{\text{UV}}$ in (6.43) in terms of rescaled variables of the order of $\mathcal{O}(1)$:

$$u = \frac{M}{3} J, \quad u' = \frac{M}{3} J', \quad u'' = \frac{M}{3} J''. \quad (\text{D.19})$$

Then $f(J', J''; J)$ becomes

$$f(J', J''; J) = \hat{C} \left(\frac{M}{3} \right)^{-(N_1+N_2)-N_J-N_\Delta} (u' - u'')^{N_\Delta} (u')^{N_1} (u'')^{N_2}, \quad (\hat{C} = C u^{N_J}). \quad (\text{D.20})$$

Furthermore, by noting $J, J', J'' \ll j$, the Wigner $6j$ symbol can be approximated as

$$\begin{aligned} \left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 &\simeq \frac{1 + (-1)^{J'+J''+J}}{2} \frac{1}{n} \frac{\left[\left(\frac{J+J'+J''}{2} \right)! \right]^2}{\left[\left(\frac{-J+J'+J''}{2} \right)! \left(\frac{J-J'+J''}{2} \right)! \left(\frac{J+J'-J''}{2} \right)! \right]^2} \\ &\times \frac{(J+J'+J'')!(J-J'+J'')!(J+J'-J'')!}{(J+J'+J''+1)!} \end{aligned} \quad (\text{D.21})$$

with $n = 2j + 1$, e.g., according to eq. (4) in Chapter 9.9.1 and eq. (32) in Chapter 8.5.2 of [58]. Applying the Stirling formula to the factorials in the r.h.s.¹⁹ leads to

$$\left\{ \begin{matrix} J' & J'' & J \\ j & j & j \end{matrix} \right\}^2 \simeq \frac{2}{\pi n} \frac{1}{\sqrt{(J+J'+J'')(-J+J'+J'')(J-J'+J'')(J+J'-J'')}} \quad (\text{D.22})$$

$$\simeq \left(\frac{M}{3} \right)^2 \frac{2}{\pi n} \frac{1}{\sqrt{[(u' + u'')^2 - u^2][u^2 - (u' - u'')^2]}} \quad (\text{D.23})$$

for $J' + J'' + J$ even.

In the meantime, we assume that $a, b \geq 1$. The other cases, $a = 0$ or $b = 0$, will be considered separately later. The integral $I_{a,b}$ in (D.2) becomes²⁰

$$\begin{aligned} I_{a,b} &\sim \left(\frac{M}{3} \right)^{2(a+b)-n_p-n_q-2} \int \frac{d^2 q}{(2\pi)^2} \frac{g(p, q)}{(q^2 + (u')^2)^a ((q+p)^2 + (u'')^2)^b} \\ &= \left(\frac{M}{3} \right)^{2(a+b)-n_p-n_q-2} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dt t^{b-1} (1-t)^{a-1} \int \frac{d^2 q}{(2\pi)^2} \frac{g(p, q)}{[(q+tp)^2 + X(t)]^{a+b}} \\ &= \left(\frac{M}{3} \right)^{2(a+b)-n_p-n_q-2} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dt t^{b-1} (1-t)^{a-1} \int \frac{d^2 q}{(2\pi)^2} \frac{g(p, q-tp)}{[q^2 + X(t)]^{a+b}}, \end{aligned} \quad (\text{D.24})$$

¹⁹ Since $J, J', J'' \geq \mathcal{O}(M^{-1}) \gg 1$, this is justified for a generic point in Region I except for in the vicinity of the boundary. We can see that the contribution from such exceptional points is negligible in the Moyal limit.

²⁰ Note that $g(\tilde{p}, \tilde{q}) = \left(\frac{M}{3} \right)^{-n_p-n_q} g(p, q)$ as g is a homogeneous polynomial.

where we have used the Feynman integral formula,

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dt \frac{t^{a-1}(1-t)^{b-1}}{(At + B(1-t))^{a+b}}, \quad (\text{D.25})$$

and

$$X(t) \equiv t(1-t)p^2 + (1-t)(u')^2 + t(u'')^2. \quad (\text{D.26})$$

Combining (D.20), (D.23), and (D.24), we have

$$\begin{aligned} \mathcal{A}_I^{\text{UV}} &= C_I \left(\frac{M}{3} \right)^{-(N_1+N_2)-N_J-N_\Delta+2(a+b)-n_p-n_q} \\ &\times \int_{D_u} du' du'' \frac{(u')^{N_1} (u'')^{N_2} (u' - u'')^{N_\Delta}}{\sqrt{[(u' + u'')^2 - u^2] [u^2 - (u' - u'')^2]}} \\ &\times \int_0^1 dt t^{b-1} (1-t)^{a-1} \int \frac{d^2 q}{(2\pi)^2} \frac{g(p, q - tp)}{[q^2 + X(t)]^{a+b}}, \end{aligned} \quad (\text{D.27})$$

where the integration region D_u stands for

$$D_u \equiv \{(u', u'') | u \leq u' + u'' \leq u_B, |u' - u''| \leq u\} \quad (\text{D.28})$$

with $u_B \equiv \frac{M}{3} J_B = \mathcal{O}(M^{1-2\alpha}) \gg 1$, and C_I is an $\mathcal{O}(1)$ constant.

Note that the exponent of $\frac{M}{3}$ is nothing but W^- and is always positive. In fact,

$w > 0$: From Theorem 1, $W^+ > 0$ and $W^- = W^+ + w > W^+ > 0$;

$w \leq 0$: $W^- > 0$ from Theorem 1.

Similarly to the case of Theorem 1, let us consider the situation that the q -integrals in $I_{a,b}$ converge in the UV region, because the assumption of Theorem 2, that $\mathcal{A}_{II}^{\text{UV}}$ vanishes, implies that there is no UV divergence in $\mathcal{A}_I^{\text{UV}}$. The only possible divergence in the integration in (D.27) is from the IR region: $u' \sim 0$ or $u'' \sim 0$.

Note that $g(p, q - tp)$ can be expanded as

$$g(p, q - tp) = \sum_{\ell \geq 0} \alpha_\ell(t) q^{2\ell}, \quad (\text{D.29})$$

because parity-odd terms with respect to $q_\mu \rightarrow -q_\mu$ trivially vanish in the integration. The coefficient $\alpha_\ell(t)$ that depends on p^2 is a polynomial of t . After the rescaling $q_\mu = X(t)^{1/2} \hat{q}_\mu$, we obtain

$$\mathcal{A}_I^{\text{UV}} \sim \left(\frac{M}{3} \right)^{W^-} \sum_{\ell \geq 0} \int \frac{d^2 \hat{q}}{(2\pi)^2} \frac{\hat{q}^{2\ell}}{(\hat{q}^2 + 1)^{a+b}}$$

$$\times \int_{D_u} du' du'' \frac{(u')^{N_1} (u'')^{N_2} (u' - u'')^{N_\Delta}}{\sqrt{[(u' + u'')^2 - u^2] [u^2 - (u' - u'')^2]}} I_\ell(u', u''), \quad (\text{D.30})$$

$$I_\ell(u', u'') \equiv \int_0^1 dt \frac{t^{b-1} (1-t)^{a-1} \alpha_\ell(t)}{[t(1-t)p^2 + (1-t)(u')^2 + t(u'')^2]^{a+b-\ell-1}}. \quad (\text{D.31})$$

As mentioned above, we are considering the situation that the \hat{q} -integrals UV converge, i.e.

$$a + b - \ell - 1 > 0 \quad (\text{D.32})$$

for each ℓ , and $a, b \geq 1$ in the computation here.

For arbitrary ℓ such that $\alpha_\ell(t) \neq 0$, the most singular behaviors as $u' \sim 0$ and $u'' \sim 0$ come from $t \sim 0$ and $t \sim 1$ in the integral $I_\ell(u', u'')$, respectively. We evaluate them with the assumption that $\alpha_\ell(t)$ behaves as (6.48). For $u' \sim 0$, the dominant contributions to I_ℓ are

$$\begin{aligned} I_\ell(u' \sim 0, u'') &\sim \int_0^{(u')^2} dt \frac{t^{b-1+a_\ell^{(0)}}}{[(u')^2 + t((u'')^2 + p^2 - (u')^2) - t^2 p^2]^{a+b-\ell-1}} \\ &= (u')^{-2(a-\ell-a_\ell^{(0)}-1)} \int_0^1 dt \frac{t^{b-1+a_\ell^{(0)}}}{[1 + t((u'')^2 + p^2 - (u')^2) - t^2 (u')^2 p^2]^{a+b-\ell-1}} \\ &\sim (u')^{-2(a-\ell-a_\ell^{(0)}-1)}. \end{aligned} \quad (\text{D.33})$$

The integral in the second line is finite since $u'' \gtrsim u = \mathcal{O}(1)$ for $u' \sim 0$ in D_u . Similarly, we have

$$I_\ell(u', u'' \sim 0) \sim (u'')^{-2(b-\ell-a_\ell^{(1)}-1)}. \quad (\text{D.34})$$

The IR contribution in $\mathcal{A}_I^{\text{UV}}$ around $(u', u'') = (0, u)$ comes from (D.33):

$$\mathcal{A}_{I(0,u)}^{\text{UV}} \equiv M^{W-} \int_{(0,u)} du' du'' \frac{(u')^{N_1-2(a-\ell-a_\ell^{(0)}-1)}}{\sqrt{[(u' + u'')^2 - u^2] [u^2 - (u' - u'')^2]}}, \quad (\text{D.35})$$

which is evaluated by changing the variables as

$$u' = x \cos \theta, \quad u'' = u + x \sin \theta \quad (\text{D.36})$$

with $x \gtrsim \frac{M}{3}$ and $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then,

$$\begin{aligned} \mathcal{A}_{I(0,u)}^{\text{UV}} &= M^{W-} \left(\int_{M/3} dx x^{N_1-2(a-\ell-a_\ell^{(0)}-1)} \right) \frac{1}{u} \int_0^{\pi/4} d\theta \frac{(\cos \theta)^{N_1-2(a-\ell-a_\ell^{(0)}-1)}}{\sqrt{\cos(2\theta)}} \\ &\sim M^{W-+N_1-2(a-\ell-a_\ell^{(0)}-1)+1}. \end{aligned} \quad (\text{D.37})$$

Note that the θ -integral is finite. The remaining IR contribution around $(u', u'') = (u, 0)$,

$$\mathcal{A}_{I(u,0)}^{\text{UV}} \equiv M^{W^-} \int_{(u,0)} du' du'' \frac{(u'')^{N_2-2(b-\ell-a_\ell^{(1)}-1)}}{\sqrt{[(u'+u'')^2-u^2][u^2-(u'-u'')^2]}}, \quad (\text{D.38})$$

can be obtained in the same manner as

$$\mathcal{A}_{I(u,0)}^{\text{UV}} \sim M^{W^-+N_2-2(b-\ell-a_\ell^{(1)}-1)+1}. \quad (\text{D.39})$$

From (D.37) and (D.39), we can see that $\mathcal{A}_I^{\text{UV}}$ vanishes in the Moyal limit if both

$$W^- + \left(N_1 - 2(a - \ell - a_\ell^{(0)} - 1) \right) + 1 \quad (\text{D.40})$$

and

$$W^- + \left(N_2 - 2(b - \ell - a_\ell^{(1)} - 1) \right) + 1 \quad (\text{D.41})$$

are positive for arbitrary ℓ such that $\alpha_\ell(t) \neq 0$. This is equivalent to the statement of Theorem 2 for $a, b \geq 1$.

Next, let us consider the case that one of a and b is zero (say $b = 0$). Without using (D.25), we have

$$\begin{aligned} \mathcal{A}_I^{\text{UV}}|_{b=0} &= C_I \left(\frac{M}{3} \right)^{W^-} \int_{D_u} du' du'' \frac{(u')^{N_1} (u'')^{N_2} (u' - u'')^{N_\Delta}}{\sqrt{[(u'+u'')^2-u^2][u^2-(u'-u'')^2]}} \\ &\quad \times \int \frac{d^2 q}{(2\pi)^2} \frac{g(p, q)}{[q^2 + (u')^2]^a}. \end{aligned} \quad (\text{D.42})$$

The rescaling $q_\mu = u' \hat{q}_\mu$ and the expansion (D.29) with $t = 0$ lead to

$$\begin{aligned} \mathcal{A}_I^{\text{UV}}|_{b=0} &\sim \left(\frac{M}{3} \right)^{W^-} \sum_{\ell \geq 0} \alpha_\ell(0) \int \frac{d^2 \hat{q}}{(2\pi)^2} \frac{\hat{q}^{2\ell}}{(\hat{q}^2 + 1)^a} \\ &\quad \times \int_{D_u} du' du'' \frac{(u')^{N_1+2(-a+\ell+1)} (u'')^{N_2} (u' - u'')^{N_\Delta}}{\sqrt{[(u'+u'')^2-u^2][u^2-(u'-u'')^2]}}. \end{aligned} \quad (\text{D.43})$$

Here, IR singular behavior around $(u', u'') = (0, u)$ has the same form as (D.35) with $a_\ell^{(0)} = 0$:

$$\begin{aligned} \mathcal{A}_{I(0,u)}^{\text{UV}}|_{b=0} &= M^{W^-} \int_{(0,u)} du' du'' \frac{(u')^{N_1-2(a-\ell-1)}}{\sqrt{[(u'+u'')^2-u^2][u^2-(u'-u'')^2]}} \\ &\sim M^{W^-+N_1-2(a-\ell-1)+1}, \end{aligned} \quad (\text{D.44})$$

while the contribution around $(u', u'') = (u, 0)$ becomes proportional to $M^{W^-+N_2+1}$, which clearly vanishes as $M \rightarrow 0$. This proves Theorem 2 for the $b = 0$ case.

E Flat limit of fuzzy spherical harmonics

In this appendix, we express the fuzzy spherical harmonics in terms of plane waves on the Moyal plane in order to prepare to obtain the one-loop effective action in the Moyal limit (the limit at Step 1 in section 4). Once the expression is obtained, taking the commutative limit (at Step 2) is straightforward.

E.1 $|j\ r\rangle$ near $r = j$ in large j

Let us consider the spin- (j, r) eigenstates of the $SU(2)$ generators \vec{L} given in (B.1) and (B.2). For large j , we focus on a region near the north pole ($r = j$) to put

$$r = j - s \quad (s = 0, 1, 2, \dots) \quad \text{with} \quad s \ll \mathcal{O}(j). \quad (\text{E.1})$$

Then, the second and third equations in (B.2) become

$$\begin{aligned} L_+ |j\ j - s\rangle &= \sqrt{2j}\sqrt{s} [1 + \mathcal{O}(s/j)] |j\ j - s + 1\rangle, \\ L_- |j\ j - s\rangle &= \sqrt{2j}\sqrt{s+1} [1 + \mathcal{O}(s/j)] |j\ j - s - 1\rangle, \end{aligned} \quad (\text{E.2})$$

which suggests the definition

$$|s\rangle\rangle \equiv |j\ j - s\rangle, \quad a \equiv \frac{1}{\sqrt{2j}} L_+, \quad a^\dagger \equiv \frac{1}{\sqrt{2j}} L_- \quad (\text{E.3})$$

so that a (a^\dagger) acts on $|s\rangle\rangle$ as an annihilation (creation) operator:

$$a |s\rangle\rangle = \sqrt{s} |s-1\rangle\rangle, \quad a^\dagger |s\rangle\rangle = \sqrt{s+1} |s+1\rangle\rangle, \quad (\text{E.4})$$

satisfying the algebra

$$[a, a^\dagger] = 1 \quad (\text{E.5})$$

on the Hilbert space spanned by $|s\rangle\rangle$. These operators a and a^\dagger can be regarded as coordinates of the Moyal plane $(\hat{\xi}, \hat{\eta})$:

$$[\hat{\xi}, \hat{\eta}] = i\Theta \quad (\text{E.6})$$

with

$$\hat{\zeta} \equiv \hat{\xi} + i\hat{\eta} = \sqrt{2\Theta} a, \quad \hat{\zeta}^\dagger \equiv \hat{\xi} - i\hat{\eta} = \sqrt{2\Theta} a^\dagger. \quad (\text{E.7})$$

E.2 $\hat{Y}_{Jm}^{(jj)}$ in large j

Let us substitute $r = j - s$, $r' = j - s'$ in the definition of $\hat{Y}_{Jm}^{(jj)}$ (B.7) and consider the case $s, s' \ll \mathcal{O}(j)$:

$$\hat{Y}_{Jm}^{(jj)} = \sqrt{2J+1} \sum_{s,s'} C_{jj-s' Jm}^{jj-s} |s\rangle \langle s'|. \quad (\text{E.8})$$

Since the C-G coefficient $C_{c\alpha b\beta}^{c\gamma}$ for $c \gg b$ can be evaluated as ²¹

$$C_{c\alpha b\beta}^{c\gamma} \simeq \delta_{\beta, \gamma-\alpha} \sqrt{\frac{4\pi}{2b+1}} Y_{b\beta}(\vartheta, 0) \quad (\text{E.9})$$

with $Y_{b\beta}$ being a spherical harmonics and

$$\cos \frac{\vartheta}{2} = \sqrt{\frac{c + \gamma + \frac{1}{2}}{2c+1}}, \quad \sin \frac{\vartheta}{2} = \sqrt{\frac{c - \gamma + \frac{1}{2}}{2c+1}}, \quad (\text{E.10})$$

we have

$$C_{jj-s' Jm}^{jj-s} \simeq \delta_{m, s'-s} \sqrt{\frac{4\pi}{2J+1}} Y_{Jm}(\vartheta, 0) \quad (\text{E.11})$$

with

$$\cos \frac{\vartheta}{2} = \sqrt{1 - \frac{s + \frac{1}{2}}{n}}, \quad \sin \frac{\vartheta}{2} = \sqrt{\frac{s + \frac{1}{2}}{n}}. \quad (\text{E.12})$$

Note that $\vartheta \simeq 2\sqrt{\frac{s+\frac{1}{2}}{n}}$ is actually small for $s \ll j$, corresponding to the fact that we are looking at the neighborhood of the north pole. Then the fuzzy spherical harmonics (E.8) can be evaluated as

$$\hat{Y}_{Jm}^{(jj)} \simeq \sqrt{4\pi} \sum_{s,s'} \delta_{m, s'-s} Y_{Jm}(\vartheta, 0) |s\rangle \langle s'|. \quad (\text{E.13})$$

E.3 Stereographic transformation

Next, we consider a stereographic transformation from $S^2 \setminus (\text{south pole})$ with the radius $R = 3/M$ to \mathbb{R}^2 . As in Fig. 2, the plane \mathbb{R}^2 is tangent to the S^2 at the north pole, and the north pole coincides with the origin of \mathbb{R}^2 . The coordinates (ϑ, φ) of S^2 are mapped to the coordinates (ξ, η) of \mathbb{R}^2 as

$$\xi = 2R \tan \frac{\vartheta}{2} \cos \varphi, \quad \eta = 2R \tan \frac{\vartheta}{2} \sin \varphi. \quad (\text{E.14})$$

Equivalently, in terms of the complex coordinates $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$,

$$\zeta = 2R \tan \frac{\vartheta}{2} e^{i\varphi}, \quad \bar{\zeta} = 2R \tan \frac{\vartheta}{2} e^{-i\varphi}. \quad (\text{E.15})$$

²¹See, e.g., eq. (7) in section 8.9.1 of [58].

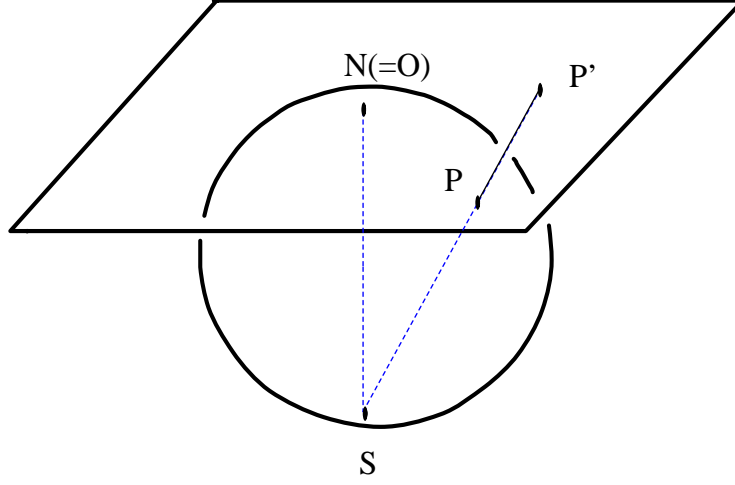


Figure 2: The stereographic transformation from $S^2 \setminus (\text{south pole})$ to \mathbb{R}^2 . The ray from the south pole (S) maps the point P on S^2 to the point P' on \mathbb{R}^2 . The north pole (N) coincides with the origin (O) on the plane.

In the Moyal limit, we send $R \rightarrow \infty$ and $\vartheta \rightarrow 0$ with $R\vartheta$ kept fixed. In this limit, ζ and $\bar{\zeta}$ become

$$\zeta \rightarrow R\vartheta e^{i\varphi}, \quad \bar{\zeta} \rightarrow R\vartheta e^{-i\varphi}. \quad (\text{E.16})$$

Recall that the radius R and the noncommutativity of the Moyal plane are given by $R = \frac{3}{M}$ and $\Theta = \frac{18}{M^2 n}$, respectively. Then, when ϑ is small as $\vartheta \simeq 2\sqrt{\frac{s+\frac{1}{2}}{n}}$, the absolute value of ζ can be written as

$$|\zeta| = R\vartheta = \frac{6}{M\sqrt{n}} \sqrt{s + \frac{1}{2}} = \sqrt{2\Theta} \sqrt{s + \frac{1}{2}}, \quad (\text{E.17})$$

which corresponds to the “radial length” of the Moyal coordinates (E.7):

$$\frac{1}{2}(\hat{\zeta}\hat{\zeta}^\dagger + \hat{\zeta}^\dagger\hat{\zeta}) = \Theta(aa^\dagger + a^\dagger a) = 2\Theta \left(s + \frac{1}{2} \right) \quad \text{on} \quad |s\rangle. \quad (\text{E.18})$$

Let us next rewrite the spherical harmonics in (E.13) using these parametrizations. When $J \rightarrow \infty$ and $\vartheta \rightarrow 0$ with finite $J\vartheta$, the harmonics $Y_{Jm}(\vartheta, 0)$ can be expressed by the Bessel function as ²²

$$Y_{Jm}(\vartheta, 0) \simeq \sqrt{\frac{J}{2\pi}} (-1)^m J_m(J\vartheta). \quad (\text{E.19})$$

²²For example, see eq. (9) in section 5.12.3 of [58].

In our case, since the absolute value of the momentum in the \mathbb{R}^2 is kept finite as

$$u = \frac{1}{R}J = \frac{M}{3}J, \quad (\text{E.20})$$

the combination $J\vartheta$ is evaluated as

$$J\vartheta = uR\vartheta = u\sqrt{2\Theta}\sqrt{s + \frac{1}{2}} = u|\zeta|. \quad (\text{E.21})$$

Then (E.13) can be rewritten as

$$\hat{Y}_{Jm}^{(jj)} \simeq \sqrt{\frac{6u}{M}} \sum_{s,s'} \delta_{m,s'-s} (-1)^m J_m(u|\zeta|) |s\rangle \langle s'|. \quad (\text{E.22})$$

More explicitly,

$$\hat{Y}_{J0}^{(jj)} \simeq \sqrt{\frac{6u}{M}} \sum_{s=0,1,2,\dots} J_0 \left(u\sqrt{2\Theta}\sqrt{s + \frac{1}{2}} \right) |s\rangle \langle s|, \quad (\text{E.23})$$

and, for $m = 1, 2, \dots$,

$$\hat{Y}_{Jm}^{(jj)} \simeq \sqrt{\frac{6u}{M}} (-1)^m \sum_{s=0,1,2,\dots} J_m \left(u\sqrt{2\Theta}\sqrt{s + \frac{1}{2}} \right) |s\rangle \langle m+s|, \quad (\text{E.24})$$

$$\begin{aligned} \hat{Y}_{J-m}^{(jj)} &= (-1)^{-m} \left(\hat{Y}_{Jm}^{(jj)} \right)^\dagger \\ &\simeq \sqrt{\frac{6u}{M}} \sum_{s=0,1,2,\dots} J_m \left(u\sqrt{2\Theta}\sqrt{s + \frac{1}{2}} \right) |m+s\rangle \langle s|. \end{aligned} \quad (\text{E.25})$$

Here we should note that eq. (E.19) holds even for J smaller than $\mathcal{O}(1/M)$. More precisely, it is valid for $J \geq \mathcal{O}(M^{-\varepsilon})$ with $0 < \varepsilon \ll 1$. In fact, according to the formula (4) in section 5.12.2 of [58], $Y_{J\pm m}(\vartheta, 0)$ can be evaluated as

$$Y_{J\pm m}(\vartheta, 0) \simeq (\mp 1)^m \frac{1}{m!} \sqrt{\frac{2J+1}{4\pi}} \frac{(J+m)!}{(J-m)!} \left(\frac{\vartheta}{2} \right)^m \left[1 - \frac{3J(J+1) - m(m+1)}{3(m+1)} \left(\frac{\vartheta}{2} \right)^2 \right] \quad (\text{E.26})$$

for $0 < \vartheta \ll 1$ and $m = 0, 1, \dots, J$. When $J \gg \mathcal{O}(1)$ and $J\vartheta \ll \mathcal{O}(1)$, it becomes

$$\begin{aligned} Y_{J\pm m}(\vartheta, 0) &\simeq (\mp 1)^m \frac{1}{m!} \sqrt{\frac{J}{2\pi}} \left(\frac{J\vartheta}{2} \right)^m \left[1 - \frac{1}{m+1} \left(\frac{J\vartheta}{2} \right)^2 \right] \\ &= (\mp 1)^m \sqrt{\frac{J}{2\pi}} J_m(J\vartheta) + \mathcal{O}((J\vartheta)^{m+4}), \end{aligned} \quad (\text{E.27})$$

which reproduces the first two terms of the expansion of (E.19) for small $J\vartheta$.

Similarly, $Y_{Jm}(\vartheta, 0)$ for $J \leq \mathcal{O}(M^{-\varepsilon})$ reads

$$\begin{aligned} Y_{J0}(\vartheta, 0) &\simeq \sqrt{\frac{2J+1}{4\pi}} \times [1 + \mathcal{O}(\vartheta^2)] = \mathcal{O}(1), \\ Y_{J\pm m}(\vartheta, 0) &\simeq \mathcal{O}(\vartheta^m) = \mathcal{O}(M^m) \sim 0 \quad (m = 1, 2, \dots, J). \end{aligned} \quad (\text{E.28})$$

Plugging this into (E.13) leads to

$$\hat{Y}_{Jm}^{(jj)} \simeq \delta_{m0} \sqrt{2J+1} \times \mathbb{1}_n \quad \text{for } J \leq \mathcal{O}(M^{-\varepsilon}). \quad (\text{E.29})$$

E.4 Plane-wave basis on the Moyal plane

A field $f(x)$ on \mathbb{R}^2

$$f(x) = \int \frac{d^2 p}{(2\pi)^2} e^{ip \cdot x} \tilde{f}(p) \quad (\text{E.30})$$

with $x_\mu = (\xi, \eta)$ and $p \cdot x = p_1 \xi + p_2 \eta$ corresponds to the operator

$$\hat{f} = \int \frac{d^2 p}{(2\pi)^2} e^{ip \cdot \hat{x}} \tilde{f}(p) \quad (\text{E.31})$$

with $p \cdot \hat{x} = p_1 \hat{\xi} + p_2 \hat{\eta}$. The field $f(x)$ and the operator \hat{f} are connected with each other through the operator

$$\hat{\Delta}(x) \equiv \int \frac{d^2 p}{(2\pi)^2} e^{ip \cdot (\hat{x} - x)} \quad (\text{E.32})$$

as

$$\hat{f} = \int d^2 x f(x) \hat{\Delta}(x), \quad f(x) = 2\pi \Theta \text{Tr} \left(\hat{f} \hat{\Delta}(x) \right). \quad (\text{E.33})$$

The second formula follows from

$$\text{Tr} \left(e^{ip \cdot \hat{x}} e^{iq \cdot \hat{x}} \right) = \frac{2\pi}{\Theta} \delta^2(p + q), \quad (\text{E.34})$$

which can be shown, for instance, by using the eigenstate of $\hat{\xi}$:

$$\hat{\xi}|\xi\rangle = \xi|\xi\rangle, \quad \langle \xi | \hat{\eta} = -i\Theta \partial_\xi \langle \xi |, \quad \text{Tr}(\dots) = \int d\xi \langle \xi | (\dots) | \xi \rangle. \quad (\text{E.35})$$

It can also be shown that

$$\text{Tr} \left(\hat{\Delta}(x) \hat{\Delta}(x') \right) = \frac{1}{2\pi\Theta} \delta^2(x - x'). \quad (\text{E.36})$$

The product $\hat{f}_1 \hat{f}_2$ corresponds to the Moyal product

$$(f_1 * f_2)(x) \equiv e^{i\frac{\Theta}{2}(\partial_\xi \partial_{\eta'} - \partial_{\eta} \partial_{\xi'})} f_1(\xi, \eta) f_2(\xi', \eta') \Big|_{\xi'=\xi, \eta'=\eta} \quad (\text{E.37})$$

through

$$\hat{f}_1 \hat{f}_2 = \int d^2x (f_1 * f_2)(x) \hat{\Delta}(x), \quad (f_1 * f_2)(x) = 2\pi\Theta \text{Tr} \left(\hat{f}_1 \hat{f}_2 \hat{\Delta}(x) \right). \quad (\text{E.38})$$

From (E.33), \hat{f} can be expanded by the plane-wave basis as

$$\hat{f} = 2\pi\Theta \int d^2x \text{Tr} \left(\hat{f} \hat{\Delta}(x) \right) \hat{\Delta}(x) = 2\pi\Theta \int \frac{d^2p}{(2\pi)^2} \text{Tr} \left(\hat{f} e^{ip \cdot \hat{x}} \right) e^{-ip \cdot \hat{x}}. \quad (\text{E.39})$$

E.5 Computation of $\langle\langle s | e^{ip \cdot \hat{x}} | s' \rangle\rangle$

Let us compute the matrix elements of the plane-wave basis $e^{ip \cdot \hat{x}}$ with respect to the states $|s\rangle$. Using the complex combination

$$p \equiv p_1 + ip_2, \quad \bar{p} \equiv p_1 - ip_2, \quad (\text{E.40})$$

the basis can be expressed as

$$e^{ip \cdot \hat{x}} = e^{i\frac{1}{2}(\bar{p}\hat{\zeta} + p\hat{\zeta}^\dagger)} = e^{-\frac{\Theta}{4}p\bar{p}} e^{i\sqrt{2\Theta}\frac{p}{2}a^\dagger} e^{i\sqrt{2\Theta}\frac{\bar{p}}{2}a}. \quad (\text{E.41})$$

Then,

$$\begin{aligned} \langle\langle s | e^{ip \cdot \hat{x}} | s' \rangle\rangle &= e^{-\frac{\Theta}{4}p\bar{p}} \sum_{t=0}^s \frac{\left(i\sqrt{2\Theta}\frac{p}{2}\right)^t}{t!} \sum_{t'=0}^{s'} \frac{\left(i\sqrt{2\Theta}\frac{\bar{p}}{2}\right)^{t'}}{t'!} \langle\langle s | (a^\dagger)^t a^{t'} | s' \rangle\rangle \\ &= e^{-\frac{\Theta}{4}|p|^2} e^{i(s-s')\varphi_p} \sum_{t=0}^s \sum_{t'=0}^{s'} \delta_{t'-t, s'-s} \frac{\sqrt{s!s'!}}{(s-t)!} \frac{\left(i\sqrt{2\Theta}\frac{|p|}{2}\right)^{t+t'}}{t!t'!} \end{aligned} \quad (\text{E.42})$$

with $p = |p| e^{i\varphi_p}$ and $\bar{p} = |p| e^{-i\varphi_p}$, which leads to

$$\begin{aligned} e^{ip \cdot \hat{x}} &= \sum_{s, s'} |s\rangle \langle\langle s | e^{ip \cdot \hat{x}} | s' \rangle\rangle \langle\langle s' | \\ &= \sum_{m \in \mathbb{Z}} e^{-\frac{\Theta}{4}|p|^2} e^{-im\varphi_p} \sum_{s, s'} \delta_{m, s'-s} \sum_{t=0}^s \sum_{t'=0}^{s'} \delta_{m, t'-t} \frac{\sqrt{s!s'!}}{(s-t)!} \frac{\left(i\sqrt{2\Theta}\frac{|p|}{2}\right)^{t+t'}}{t!t'!} |s\rangle \langle\langle s' |. \end{aligned} \quad (\text{E.43})$$

Note that

$$\sum_{t=0}^s \frac{s!}{(s-t)!} \frac{\left(-\frac{\Theta}{2}|p|^2\right)^t}{(m+t)!t!} = \frac{1}{m!} F\left(-s, m+1; \frac{\Theta}{2}|p|^2\right) \quad (\text{E.44})$$

for $m = 0, 1, 2, \dots$, where $F(a, b; z)$ is a confluent hypergeometric function defined by

$$F(a, b; z) \equiv \sum_{t=0}^{\infty} \frac{(a)_t}{(b)_t} \frac{z^t}{t!} \quad (\text{E.45})$$

with $(a)_t \equiv a(a+1)\cdots(a+t-1)$ and $(a)_0 \equiv 1$. Then, (E.43) can be expressed as

$$\begin{aligned} e^{ip \cdot \hat{x}} &= e^{-\frac{\Theta}{4}|p|^2} \sum_{s=0}^{\infty} \left[F\left(-s, 1; \frac{\Theta}{2}|p|^2\right) |s\rangle\rangle \langle\langle s| \right. \\ &\quad + \sum_{m=1}^{\infty} \frac{\left(i\sqrt{2\Theta} \frac{|p|}{2}\right)^m}{m!} \sqrt{\frac{(m+s)!}{s!}} F\left(-s, m+1; \frac{\Theta}{2}|p|^2\right) \\ &\quad \left. \times \left(e^{-im\varphi_p} |s\rangle\rangle \langle\langle m+s| + e^{im\varphi_p} |m+s\rangle\rangle \langle\langle s| \right) \right]. \quad (\text{E.46}) \end{aligned}$$

E.6 $\hat{Y}_{Jm}^{(jj)}$ in terms of the plane-wave basis

By using (E.39), $\hat{Y}_{Jm}^{(jj)}$ can be expanded as

$$\hat{Y}_{Jm}^{(jj)} = 2\pi\Theta \int \frac{d^2p}{(2\pi)^2} \text{Tr} \left(\hat{Y}_{Jm}^{(jj)} e^{ip \cdot \hat{x}} \right) e^{-ip \cdot \hat{x}}. \quad (\text{E.47})$$

Plugging (E.23), (E.24), (E.25), and (E.46) into (E.47) leads to

$$\hat{Y}_{Jm}^{(jj)} = 2\pi\Theta \sqrt{\frac{6u}{M}} \int \frac{d^2p}{(2\pi)^2} e^{-\frac{\Theta}{4}|p|^2} e^{im\varphi_p} \mathcal{K}_m(\Theta, u, |p|) e^{-ip \cdot \hat{x}} \quad (\text{E.48})$$

for $J \geq \mathcal{O}(M^{-\varepsilon})$. The kernels are given by

$$\mathcal{K}_0(\Theta, u, |p|) \equiv \sum_{s=0}^{\infty} J_0 \left(u\sqrt{2\Theta} \sqrt{s + \frac{1}{2}} \right) F\left(-s, 1; \frac{\Theta}{2}|p|^2\right), \quad (\text{E.49})$$

$$\mathcal{K}_m(\Theta, u, |p|) \equiv \frac{\left(-i\sqrt{2\Theta} \frac{|p|}{2}\right)^m}{m!} \sum_{s=0}^{\infty} \sqrt{\frac{(m+s)!}{s!}} J_m \left(u\sqrt{2\Theta} \sqrt{s + \frac{1}{2}} \right) F\left(-s, m+1; \frac{\Theta}{2}|p|^2\right), \quad (\text{E.50})$$

$$\mathcal{K}_{-m}(\Theta, u, |p|) \equiv \frac{\left(i\sqrt{2\Theta} \frac{|p|}{2}\right)^m}{m!} \sum_{s=0}^{\infty} \sqrt{\frac{(m+s)!}{s!}} J_m \left(u\sqrt{2\Theta} \sqrt{s + \frac{1}{2}} \right) F\left(-s, m+1; \frac{\Theta}{2}|p|^2\right) \quad (\text{E.51})$$

for $m = 1, 2, \dots$.

E.7 Computation of the kernels

We can explicitly compute the kernels (E.49), (E.50), and (E.51) in the case of small Θ .

E.7.1 \mathcal{K}_0

It is convenient to insert

$$1 = 2 \int_0^\infty dv v v^{2s} e^{-v^2} \frac{1}{s!} \quad (\text{E.52})$$

into the sum in (E.49) for the computation. Then $\mathcal{K}_0(\Theta, u, |p|)$ becomes

$$\mathcal{K}_0(\Theta, u, |p|) = 2 \int_0^\infty dv v \sum_{s=0}^\infty v^{2s} e^{-v^2} J_0 \left(u \sqrt{2\Theta} \sqrt{s + \frac{1}{2}} \right) \sum_{t=0}^s \frac{\left(-\frac{\Theta}{2} |p|^2\right)^t}{(s-t)! (t!)^2}. \quad (\text{E.53})$$

After changing the order of the sums as

$$\sum_{s=0}^\infty \sum_{t=0}^s \cdots = \sum_{t=0}^\infty \sum_{s=t}^\infty \cdots, \quad (\text{E.54})$$

and making a shift $s \rightarrow s + t$, we obtain

$$\mathcal{K}_0(\Theta, u, |p|) = 2 \int_0^\infty dv v e^{-v^2} \sum_{t=0}^\infty \frac{\left(-\frac{\Theta}{2} |p|^2 v^2\right)^t}{(t!)^2} \sum_{s=0}^\infty J_0 \left(u \sqrt{2\Theta} \sqrt{s + t + \frac{1}{2}} \right) \frac{v^{2s}}{s!}. \quad (\text{E.55})$$

First, let us evaluate the sum over s by setting $x = 2\Theta s$ and $y = 2\Theta t$. Using the Stirling formula, we have

$$\frac{v^{2s}}{s!} \simeq \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\Theta}{x}} e^{\frac{1}{2\Theta} f(x)}, \quad f(x) \equiv 2x \ln v - x \ln \frac{x}{2\Theta} + x. \quad (\text{E.56})$$

For small Θ , the summation over s can be evaluated as an integral:

$$\sum_{s=0}^\infty J_0 \left(u \sqrt{2\Theta} \sqrt{s + t + \frac{1}{2}} \right) \frac{v^{2s}}{s!} \simeq \frac{1}{\sqrt{2\pi}} \frac{1}{2\Theta} \int_0^\infty dx J_0 \left(u \sqrt{x + y + \Theta} \right) \sqrt{\frac{2\Theta}{x}} e^{\frac{1}{2\Theta} f(x)},$$

in which contributions to the integral localize to the saddle point x_0 of $f(x)$ with

$$x_0 = 2\Theta v^2, \quad f(x_0) = 2\Theta v^2, \quad f''(x_0) = -\frac{1}{2\Theta v^2}. \quad (\text{E.57})$$

Including Gaussian fluctuations around the saddle point, we obtain

$$\sum_{s=0}^\infty J_0 \left(u \sqrt{2\Theta} \sqrt{s + t + \frac{1}{2}} \right) \frac{v^{2s}}{s!} \simeq J_0 \left(u \sqrt{\Theta} \sqrt{2v^2 + 2t + 1} \right) e^{v^2}. \quad (\text{E.58})$$

Next, the region of t giving a dominant contribution to the sum is $0 \leq t \lesssim t_0$ with t_0 satisfying

$$(t_0!)^2 = \left(\frac{\Theta}{2} |p|^2 v^2 \right)^{t_0}, \quad (\text{E.59})$$

because for $t \gtrsim t_0$, the denominator $(t!)^2$ grows much faster than the numerator, and the contribution becomes negligible. The value of t_0 is evaluated as

$$t_0 \simeq \sqrt{\frac{\Theta}{2}} |p| v, \quad (\text{E.60})$$

which means that the t -dependence of $J_0 \left(u \sqrt{\Theta} \sqrt{2v^2 + 2t + 1} \right)$ in (E.58) can be dropped and replaced with $J_0 \left(u \sqrt{\Theta} \sqrt{2v^2 + 1} \right)$ for small Θ . Noting

$$\sum_{t=0}^{\infty} \frac{\left(-\frac{\Theta}{2} |p|^2 v^2 \right)^t}{(t!)^2} = J_0 \left(\sqrt{2\Theta} |p| v \right), \quad (\text{E.61})$$

we have

$$\mathcal{K}_0(\Theta, u, |p|) \simeq 2 \int_0^{\infty} dv v J_0 \left(\sqrt{2\Theta} |p| v \right) J_0 \left(u \sqrt{\Theta} \sqrt{2v^2 + 1} \right). \quad (\text{E.62})$$

Finally, the region of v dominant to the integral is $v \gg 1$. The region $0 < v \lesssim \mathcal{O}(1)$ gives merely an $\mathcal{O}(1)$ contribution to the integral. So, in the expansion

$$J_0 \left(u \sqrt{\Theta} \sqrt{2v^2 + 1} \right) = J_0 \left(u \sqrt{2\Theta} v \right) + \frac{u \sqrt{2\Theta}}{4v} J_0' \left(u \sqrt{2\Theta} v \right) + \dots, \quad (\text{E.63})$$

we can neglect the second- and higher-order terms for $v \gg 1$. Hence,

$$\mathcal{K}_0(\Theta, u, |p|) \simeq 2 \int_0^{\infty} dv v J_0 \left(\sqrt{2\Theta} |p| v \right) J_0 \left(\sqrt{2\Theta} u v \right) = \frac{1}{\Theta} \frac{1}{|p|} \delta(|p| - u), \quad (\text{E.64})$$

where the formula

$$\int_0^{\infty} dr r J_m(\alpha r) J_m(\beta r) = \frac{1}{\alpha} \delta(\alpha - \beta) \quad (\text{E.65})$$

for $\alpha, \beta > 0$ was used. Equation (E.65) is derived below.

From (E.48) and (E.64), we obtain

$$\hat{Y}_{J_0}^{(jj)} \simeq 2\pi \sqrt{\frac{6u}{M}} e^{-\frac{\Theta}{4} u^2} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{|p|} \delta(|p| - u) e^{-ip \cdot \hat{x}} \quad (\text{E.66})$$

for $J \geq \mathcal{O}(M^{-\varepsilon})$ and small Θ .

Derivation of (E.65) Since $J_m(z) = (-1)^{-m} J_{-m}(z)$, it is sufficient to show (E.65) for $m = 0, 1, 2, \dots$. Let us start with the Bessel equations

$$\left(\partial_r^2 + \frac{1}{r} \partial_r + \alpha^2 - \frac{m^2}{r^2} \right) J_m(\alpha r) = 0, \quad (\text{E.67})$$

$$\left(\partial_r^2 + \frac{1}{r} \partial_r + \beta^2 - \frac{m^2}{r^2} \right) J_m(\beta r) = 0. \quad (\text{E.68})$$

With use of the asymptotic form

$$J_m(z) = \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{2m+1}{4} \pi \right) \times \{1 + \mathcal{O}(1/z)\}, \quad (\text{E.69})$$

$\int_0^\infty dr r [J_m(\beta r) \times (\text{E.67}) - J_m(\alpha r) \times (\text{E.68})]$ leads to

$$\begin{aligned} \int_0^\infty dr r J_m(\alpha r) J_m(\beta r) &= \frac{1}{\alpha^2 - \beta^2} \frac{1}{\pi} \lim_{r \rightarrow \infty} \left[\left(\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} \right) \sin((\alpha - \beta)r) \right. \\ &\quad \left. - \left(\sqrt{\frac{\alpha}{\beta}} - \sqrt{\frac{\beta}{\alpha}} \right) (-1)^m \cos((\alpha + \beta)r) \right]. \quad (\text{E.70}) \end{aligned}$$

Let us pick a test function $f(\beta)$ such that $f(0)$ is finite and $f(\beta)$ rapidly decays as $\beta \rightarrow \infty$, and compute $\int_0^\infty d\beta \beta f(\beta) \times (\text{E.70})$. Noting that

$$\begin{aligned} \int_0^\infty d\beta \beta f(\beta) \frac{1}{\alpha^2 - \beta^2} \left(\sqrt{\frac{\alpha}{\beta}} - \sqrt{\frac{\beta}{\alpha}} \right) \cos((\alpha + \beta)r) &= \int_0^\infty d\beta \frac{f(\beta)}{\alpha + \beta} \sqrt{\frac{\beta}{\alpha}} \cos((\alpha + \beta)r) \\ &\rightarrow 0 \quad (r \rightarrow \infty) \quad (\text{E.71}) \end{aligned}$$

due to the rapid oscillation as $r \rightarrow \infty$, we obtain

$$\int_0^\infty d\beta \beta f(\beta) \int_0^\infty dr r J_m(\alpha r) J_m(\beta r) = \lim_{r \rightarrow \infty} \frac{1}{\pi} \int_0^\infty d\beta f(\beta) \sqrt{\frac{\beta}{\alpha}} \frac{\sin((\beta - \alpha)r)}{\beta - \alpha}. \quad (\text{E.72})$$

Because the contribution to the integral on the r.h.s. localizes to the neighborhood of $\beta = \alpha$ due to the rapid oscillation,

$$\begin{aligned} (\text{r.h.s. of (E.72)}) &= \frac{f(\alpha)}{\pi} \lim_{r \rightarrow \infty} \int_0^\infty d\beta \frac{\sin((\beta - \alpha)r)}{\beta - \alpha} = \frac{f(\alpha)}{\pi} \int_{-\infty}^\infty dy \frac{\sin y}{y} = f(\alpha). \quad (\text{E.73}) \end{aligned}$$

This shows that (E.65) holds.

E.7.2 \mathcal{K}_m ($m = 1, 2, \dots$)

Following similar steps to the case of \mathcal{K}_0 , the expression for \mathcal{K}_m corresponding to (E.55) becomes

$$\begin{aligned} \mathcal{K}_m(\Theta, u, |p|) &= \left(-i\sqrt{2\Theta}\frac{|p|}{2}\right)^m 2 \int_0^\infty dv v e^{-v^2} \sum_{t=0}^\infty \frac{\left(-\frac{\Theta}{2}|p|^2 v^2\right)^t}{(m+t)! t!} \\ &\quad \times \sum_{s=0}^\infty J_m\left(u\sqrt{2\Theta}\sqrt{s+t+\frac{1}{2}}\right) \sqrt{\frac{(m+s+t)!}{(s+t)!}} \frac{v^{2s}}{s!}. \end{aligned} \quad (\text{E.74})$$

We treat the summation over s as discussed before:

$$\begin{aligned} \sum_{s=0}^\infty J_m\left(u\sqrt{2\Theta}\sqrt{s+t+\frac{1}{2}}\right) \sqrt{\frac{(m+s+t)!}{(s+t)!}} \frac{v^{2s}}{s!} \\ \simeq J_m\left(u\sqrt{\Theta}\sqrt{2v^2+2t+1}\right) (m+v^2+t)^{m/2} e^{v^2}. \end{aligned} \quad (\text{E.75})$$

Since the summation over t is dominated by the region $0 \leq t \lesssim \sqrt{\frac{\Theta}{2}}|p|v$, the t -dependence of $J_m\left(u\sqrt{\Theta}\sqrt{2v^2+2t+1}\right)$ can be discarded for small Θ . Noting

$$\left(\sqrt{2\Theta}\frac{|p|}{2}v\right)^m \sum_{t=0}^\infty \frac{\left(-\frac{\Theta}{2}|p|^2 v^2\right)^t}{(m+t)! t!} = J_m\left(\sqrt{2\Theta}|p|v\right), \quad (\text{E.76})$$

we have

$$\mathcal{K}_m(\Theta, u, |p|) \simeq (-i)^m 2 \int_0^\infty dv v J_m\left(\sqrt{2\Theta}|p|v\right) \left(1 + \frac{m}{v^2}\right)^{m/2} J_m\left(u\sqrt{\Theta}\sqrt{2v^2+1}\right). \quad (\text{E.77})$$

Because the region $v \gg 1$ dominates the integral, we can approximate as

$$J_m\left(u\sqrt{\Theta}\sqrt{2v^2+1}\right) \simeq J_m\left(u\sqrt{2\Theta}v\right), \quad \left(1 + \frac{m}{v^2}\right)^{m/2} \simeq 1 \quad (\text{E.78})$$

in evaluating the integral. Hence, use of (E.65) leads to

$$\mathcal{K}_m(\Theta, u, |p|) \simeq (-i)^m 2 \int_0^\infty dv v J_m\left(\sqrt{2\Theta}|p|v\right) J_m\left(\sqrt{2\Theta}uv\right) = (-i)^m \frac{1}{\Theta} \frac{1}{|p|} \delta(|p| - u). \quad (\text{E.79})$$

From (E.48) and (E.79), we obtain

$$\hat{Y}_{Jm}^{(jj)} \simeq 2\pi \sqrt{\frac{6u}{M}} (-i)^m e^{-\frac{\Theta}{4}u^2} \int \frac{d^2p}{(2\pi)^2} \frac{1}{|p|} \delta(|p| - u) e^{im\varphi_p} e^{-ip \cdot \hat{x}} \quad (\text{E.80})$$

for $J \geq \mathcal{O}(M^{-\epsilon})$, small Θ , and $m = 1, 2, \dots$ fixed.

E.7.3 \mathcal{K}_{-m} ($m = 1, 2, \dots$)

Since (E.51) is of the same form as (E.50) with $-i$ to i changed in the prefactor $\frac{(-i\sqrt{2\Theta}\frac{|p|}{2})^m}{m!}$, the computation for \mathcal{K}_m is repeated to obtain \mathcal{K}_{-m} . The result is

$$\mathcal{K}_{-m}(\Theta, u, |p|) \simeq i^m \frac{1}{\Theta} \frac{1}{|p|} \delta(|p| - u), \quad (\text{E.81})$$

$$\hat{Y}_{J-m}^{(jj)} \simeq 2\pi \sqrt{\frac{6u}{M}} i^m e^{-\frac{\Theta}{4}u^2} \int \frac{d^2p}{(2\pi)^2} \frac{1}{|p|} \delta(|p| - u) e^{-im\varphi_p} e^{-ip \cdot \hat{x}} \quad (\text{E.82})$$

for $J \geq \mathcal{O}(M^{-\varepsilon})$, small Θ , and $m = 1, 2, \dots$ fixed.

E.8 Summary

Summarizing the results obtained in the previous subsections, for $J \geq \mathcal{O}(M^{-\varepsilon})$, small Θ and $m \in \mathbb{Z}$ fixed, the kernel and the fuzzy spherical harmonics are evaluated as

$$\mathcal{K}_m(\Theta, u, |p|) \simeq (-i)^m \frac{1}{\Theta} \frac{1}{|p|} \delta(|p| - u) \quad (m \in \mathbb{Z}) \quad (\text{E.83})$$

and

$$\hat{Y}_{Jm}^{(jj)} \simeq 2\pi \sqrt{\frac{6u}{M}} (-i)^m e^{-\frac{\Theta}{4}u^2} \int \frac{d^2p}{(2\pi)^2} \frac{1}{|p|} \delta(|p| - u) e^{im\varphi_p} e^{-ip \cdot \hat{x}}. \quad (\text{E.84})$$

Let us consider the commutative limit $\Theta \rightarrow 0$ at the final step (Step 2 in section 4). The coordinates $\hat{x}_\mu = (\hat{\xi}, \hat{\eta})$ reduce to the c-numbers $x_\mu = (\xi, \eta)$. We call the polar angle of x_μ in \mathbb{R}^2 φ . Then,

$$p \cdot \hat{x} \rightarrow |p||\zeta| \cos(\varphi_p - \varphi) \quad (\text{E.85})$$

with $\zeta = \xi + i\eta$ and $\varphi = \arg(\zeta)$. Then (E.84) becomes

$$\hat{Y}_{Jm}^{(jj)} \rightarrow \sqrt{\frac{6u}{M}} \left[\frac{(-i)^m}{2\pi} \int_0^{2\pi} d\varphi_p e^{im\varphi_p - iu|\zeta| \cos \varphi_p} \right] e^{im\varphi} = \sqrt{\frac{6u}{M}} J_{-m}(u|\zeta|) e^{im\varphi}. \quad (\text{E.86})$$

We can see that (E.86) is consistent with the orthonormality

$$\delta_{JJ'} \delta_{mm'} = \frac{1}{n} \text{tr}_n \left\{ \left(\hat{Y}_{Jm}^{(jj)} \right)^\dagger \hat{Y}_{J'm'}^{(jj)} \right\} \quad (\text{E.87})$$

as follows. In the commutative limit,

$$\frac{1}{n} \text{tr}_n \{ \dots \} \rightarrow \int \frac{d\Omega}{4\pi} \{ \dots \}, \quad d\Omega = \sin \vartheta d\vartheta d\varphi. \quad (\text{E.88})$$

From (E.21),

$$\sin \vartheta \simeq \vartheta = \frac{1}{R} |\zeta|, \quad d\vartheta = \frac{1}{R} d|\zeta|, \quad (\text{E.89})$$

and thus

$$d\Omega \simeq \frac{1}{R^2} |\zeta| d|\zeta| d\varphi. \quad (\text{E.90})$$

Using (E.86), the r.h.s. of (E.87) reads

$$\begin{aligned} (\text{r.h.s. of (E.87)}) &\rightarrow \int \frac{d\Omega}{4\pi} \frac{6}{M} \sqrt{uu'} J_{-m}(u|\zeta|) J_{-m'}(u'|\zeta|) e^{-i(m-m')\varphi} \\ &= \frac{\sqrt{uu'}}{R} \delta_{mm'} \int_0^\infty d|\zeta| |\zeta| J_{-m}(u|\zeta|) J_{-m'}(u'|\zeta|). \end{aligned} \quad (\text{E.91})$$

$J_{-m}(z) = (-1)^m J_m(z)$ and the formula (E.65) leads to

$$(\text{r.h.s. of (E.87)}) \rightarrow \frac{1}{R} \delta_{mm'} \delta(u - u'), \quad (\text{E.92})$$

which is consistent with the l.h.s. of (E.87) because $\delta_{JJ'} \rightarrow \frac{1}{R} \delta(u - u')$ in the limit.

F Nonperturbative stability of the fuzzy sphere solution

In this appendix, to examine the nonperturbative stability of the k -coincident fuzzy S^2 solution (3.1) with (3.3), we evaluate the tunneling amplitudes from the fuzzy S^2 solution

1. to the trivial vacuum ($X_a = 0$)
2. to the $(k - 1)$ -coincident fuzzy S^2 solution (8.5)
3. to the solutions (8.6)
4. to the solutions (8.7).

F.1 Tunneling amplitude to the trivial vacuum

We assume the form of a tunneling solution to the trivial vacuum to be

$$X_a(x) = f(x) \frac{M}{3} L_a \quad (\text{F.1})$$

with $f(x) = 1$ at $x_1 = -\infty$ and $f(x) = 0$ at $x_1 = +\infty$. x_1 is regarded as Euclidean time.

Plugging (F.1) into the relevant part of the action,

$$S_{\text{inst}} = \frac{1}{g_{2d}^2} \int d^2x \text{Tr} \left[(\partial_\mu X_a)^2 + \left(i[X_8, X_9] + \frac{M}{3} X_{10} \right)^2 \right]$$

$$+ \left(i[X_9, X_{10}] + \frac{M}{3} X_8 \right)^2 + \left(i[X_{10}, X_8] + \frac{M}{3} X_9 \right)^2 \Big], \quad (\text{F.2})$$

leads to

$$S_{\text{inst}} = \frac{k}{g_{2d}^2} j(j+1)n \int d^2x \left[\frac{M^2}{9} (\partial_\mu f(x))^2 + \left(\frac{M^2}{9} \right)^2 f(x)^2 (f(x) - 1)^2 \right]. \quad (\text{F.3})$$

Here,

$$\text{Tr} (L_a^2) = \text{tr}_n ((L_a^{(n)})^2) \text{tr}_k (\mathbb{1}_k) = k \text{tr}_n (j(j+1) \mathbb{1}_n) = kj(j+1)n \quad (\text{F.4})$$

has been used.

By rescaling $\xi_\mu = \frac{M}{3} x_\mu$ with $f(x) = \tilde{f}(\xi)$, the expression becomes

$$S_{\text{inst}} = \frac{k}{g_{2d}^2} j(j+1)n \frac{M^2}{9} \tilde{S}_{\text{inst}} = \frac{k}{g_{2d}^2} \frac{2}{\Theta} j(j+1) \tilde{S}_{\text{inst}}, \quad (\text{F.5})$$

$$\tilde{S}_{\text{inst}} \equiv \int d^2\xi \left[(\partial_\mu \tilde{f}(\xi))^2 + \tilde{f}(\xi)^2 (\tilde{f}(\xi) - 1)^2 \right]. \quad (\text{F.6})$$

An x_2 -independent solution (the so-called kink solution) is explicitly given by

$$\tilde{f}(\xi) = \frac{1}{2} \left[1 - \tanh \left(\frac{1}{2} \xi_1 + c \right) \right], \quad (\text{F.7})$$

where c is an integration constant.

Since the value of \tilde{S}_{inst} at the solution (F.7) is proportional to the length of the ξ_2 -direction, the tunneling rate ($\simeq e^{-S_{\text{inst}}}$) vanishes when the ξ_2 -direction is noncompact. In order to discuss this in more detail, suppose the extent of the ξ_2 -direction is $0 \leq \xi_2 \leq L$; namely, x_2 ranges over $0 \leq x_2 \leq \tilde{L} (\equiv \frac{3}{M} L)$. Then, the value of the action at the kink solution becomes

$$S_{\text{inst}} = \frac{k}{g_{2d}^2} \frac{2}{\Theta} \frac{L}{3} j(j+1) = \frac{4\pi k}{g_{4d}^2} \frac{\tilde{L}}{9} M j(j+1) \quad (\text{F.8})$$

with $g_{4d}^2 = 2\pi\Theta g_{2d}^2$. At Step 1 (the Moyal limit) in the successive limits, which sends j as $j \propto M^{-2} \rightarrow \infty$, the tunneling is completely suppressed ($S_{\text{inst}} \rightarrow +\infty$) even for \tilde{L} finite.

F.2 Tunneling amplitude to the $(k-1)$ -coincident fuzzy S^2 solution

We obtain a tunneling solution to the $(k-1)$ -coincident fuzzy S^2 solution (8.5) by assuming the form

$$X_a = \frac{M}{3} \begin{pmatrix} L_a^{(n)} \otimes \mathbb{1}_{k-1} \\ f(x) L_a^{(n)} \end{pmatrix} \quad (\text{F.9})$$

with $f(x) = 1$ at $x_1 = -\infty$ and $f(x) = 0$ at $x_1 = +\infty$.

The computation is parallel with the previous subsection. The result is given by replacing the overall factor k with 1 in (F.5) and (F.8). Thus, this tunneling is also completely suppressed at Step 1 in the successive limits.

F.3 Tunneling amplitude to the solutions (8.6)

In the process to (8.6), two of the $k L_a^{(n)}$ in the fuzzy sphere solution (3.1), (3.3) recombine to $L_a^{(n+\ell)}$ and $L_a^{(n-\ell)}$. The process for $\ell = 1$ seems the most relevant in all of the tunneling processes and should be considered. We make a bound for the amplitude, since obtaining the explicit solution seems technically complicated. As discussed in [62], we recast the action as

$$\begin{aligned} S_{\text{inst}} &= \frac{1}{g_{2d}^2} \int d^2x \text{Tr} \left[\left(\pm \partial_1 X_a + i \epsilon_{abc} X_b X_c + \frac{M}{3} X_a \right)^2 + (\partial_2 X_a)^2 \right. \\ &\quad \left. \mp \partial_1 \left\{ \frac{2}{3} i \epsilon_{abc} X_a X_b X_c + \frac{M}{3} X_a X_a \right\} \right] \\ &\geq \mp \frac{1}{g_{2d}^2} \int dx_2 \text{Tr} \left[\frac{2}{3} i \epsilon_{abc} X_a X_b X_c + \frac{M}{3} X_a X_a \right]_{x_1=-\infty}^{x_1=+\infty}, \end{aligned} \quad (\text{F.10})$$

where the signs are chosen so that the right-hand side is nonnegative. The inequality is saturated by the solution of ²³

$$\pm \partial_1 X_a + i \epsilon_{abc} X_b X_c + \frac{M}{3} X_a = 0, \quad \partial_2 X_a = 0 \quad (\text{F.11})$$

with (3.1) at $x_1 = -\infty$ and (8.6) at $x_2 = +\infty$. The right-hand side of (F.10) becomes

$$\mp \frac{\tilde{L}}{g_{2d}^2} \left(\frac{M}{3} \right)^3 \text{Tr} \frac{1}{3} \{ (L_a'')^2 - (L_a)^2 \} = \mp \frac{\tilde{L}}{g_{2d}^2} \left(\frac{M}{3} \right)^3 \frac{n}{2} \ell^2. \quad (\text{F.12})$$

Taking the lower sign, we eventually have the bound

$$S_{\text{inst}} \geq \frac{2\pi}{g_{4d}^2} \frac{\tilde{L}M}{3} \ell^2. \quad (\text{F.13})$$

Thus, we see that sending \tilde{L} to infinity faster than $1/M$ suppresses the tunneling rate and stabilizes the fuzzy sphere configuration (3.1) with (3.3).

This should be regarded as a sufficient condition, and there could be cases in which tunneling does not occur in milder conditions. For instance, suppose that tunneling solutions break Q_+ or Q_- supersymmetry. Then, zero-modes of the associated Nambu-Goldstone fermions appear in the path integral, and annihilate amplitudes that do not soak up the zero-modes.

²³Interestingly, it can be seen that the solutions in appendices F.1 and F.2 saturate the bound.

F.4 Tunneling amplitude to the solutions (8.7)

We can evaluate a similar bound for the process in which one of the k $L_a^{(n)}$ in the fuzzy sphere solution splits into $L_a^{(\ell)}$ and $L_a^{(n-\ell)}$ with $\ell = 1, 2, \dots, n-1$. Then, the upper sign is taken in (F.10). The result reads

$$S_{\text{inst}} \geq \frac{2\pi}{g_{4d}^2} \frac{\tilde{L}M}{3} \frac{\ell(n-\ell)}{2}. \quad (\text{F.14})$$

The same procedure of sending \tilde{L} as in appendix F.3 maintains the stability.

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